

EXTREMAL HOLOMORPHIC MAPS AND THE SYMMETRISED BIDISC

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ABSTRACT. We introduce the class of n -extremal holomorphic maps, a class that generalises both finite Blaschke products and complex geodesics, and apply the notion to the finite interpolation problem for analytic functions from the open unit disc into the symmetrised bidisc Γ . We show that a well-known necessary condition for the solvability of such an interpolation problem is not sufficient whenever the number of interpolation nodes is 3 or greater. We introduce a sequence $\mathcal{C}_\nu, \nu \geq 0$, of necessary conditions for solvability, prove that they are of strictly increasing strength and show that \mathcal{C}_{n-3} is insufficient for the solvability of an n -point problem for $n \geq 3$. We propose the conjecture that condition \mathcal{C}_{n-2} is necessary and sufficient for the solvability of an n -point interpolation problem for Γ and we explore the implications of this conjecture.

We introduce a classification of rational Γ -inner functions, that is, analytic functions from the disc into Γ whose radial limits at almost all points on the unit circle lie in the distinguished boundary of Γ . The classes are related to n -extremality and the conditions \mathcal{C}_ν ; we prove numerous strict inclusions between the classes.

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1. INTRODUCTION

One result in G. Pick's seminal paper [29] of 1916 states that finite Blaschke products of prescribed degree are characterised by a certain extremal property. Let us say that an analytic self-map h of the open unit disc \mathbb{D} is *n-extremal* if for every n -point subset Λ of \mathbb{D} , there do not exist an $r > 1$ and a holomorphic function $f : r\mathbb{D} \rightarrow \mathbb{D}$ such that f and h agree on Λ . A version of Pick's result can be formulated: the n -extremal holomorphic self-maps of \mathbb{D} are precisely the Blaschke products of degree at most $n - 1$.

A similar notion of extremality, but with n equal to 2, occurs in the theory of hyperbolic complex spaces introduced by S. Kobayashi and described in his book [22]. In this context one studies the geometry and function theory of a domain $\Omega \subset \mathbb{C}^d$ with the aid of 2-extremal holomorphic maps from \mathbb{D} to Ω . The notion of n -extremal map makes sense, however, in much greater generality. We consider it for elements of the space $\text{Hol}(\Omega, E)$ of holomorphic maps from a domain Ω to a subset E of \mathbb{C}^N (Definition 2.1).

A prominent theme in hyperbolic complex geometry is a kind of duality between $\text{Hol}(\mathbb{D}, \Omega)$ and $\text{Hol}(\Omega, \mathbb{D})$, typified by the celebrated theorem of L. Lempert [24], which in our terminology asserts that if Ω is convex then every 2-extremal map belonging to $\text{Hol}(\mathbb{D}, \Omega)$ is a complex geodesic of Ω (that is, has an analytic left inverse). This duality plays a role in the present paper too.

Since n -extremal maps simultaneously generalise both Blaschke products of prescribed degree and complex geodesics, they surely constitute a significant class. We have encountered them in attempting to solve a certain interpolation problem, a special case of the *μ -synthesis problem*, which arises in control engineering. We make some remarks about this application in Section 14 below; see also [10, 33] for more on this topic. The problem led us to investigate [2, 4] a special domain in \mathbb{C}^2 , the *symmetrised bidisc*, defined to be the set

$$\mathbb{G} \stackrel{\text{def}}{=} \{(z + w, zw) : z, w \in \mathbb{D}\}.$$

The rich and surprising geometry of this domain and its higher-dimensional analogues has subsequently been elaborated by many authors (for example [12, 18, 20, 26, 27]).

In this paper we focus on the finite interpolation problem for $\text{Hol}(\mathbb{D}, \Gamma)$, where Γ is the closure of \mathbb{G} : given n interpolation nodes in \mathbb{D} and n target points in Γ , we wish to determine whether there exists a function in $\text{Hol}(\mathbb{D}, \Gamma)$ that satisfies the corresponding interpolation conditions. If Γ were replaced by the closure of a Carathéodory domain then the beautiful and far-reaching classical Nevanlinna-Pick theory would apply, but the domain \mathbb{G} is inhomogeneous and at present there is no satisfactory criterion for the solvability of interpolation problems in $\text{Hol}(\mathbb{D}, \Gamma)$. There is a well-known necessary condition, which we call \mathcal{C}_0 , and which is numerically

practicable for modest n (one must check for positivity a one-parameter family of Hermitian $n \times n$ matrices). Condition \mathcal{C}_0 is also sufficient for solvability when $n = 2$; one of our principal results is that condition \mathcal{C}_0 is *not* sufficient for solvability if $n \geq 3$. Accordingly, we introduce a sequence of necessary conditions \mathcal{C}_ν , for $\nu = 0, 1, 2, \dots$ (Definition 4.2). We prove that this sequence of conditions is of strictly increasing strength, from which it follows that none of the \mathcal{C}_ν is sufficient for all finite interpolation problems in $\text{Hol}(\mathbb{D}, \Gamma)$. Nevertheless, it seems possible that \mathcal{C}_{n-2} is sufficient for the solvability of an n -point problem: we conjecture that this is indeed so (the Γ -interpolation Conjecture, Section 4), and we explore the consequences of this conjecture.

To obtain our results we analyse Γ -inner functions: these are maps $h \in \text{Hol}(\mathbb{D}, \Gamma)$ whose radial limits almost everywhere on the unit circle \mathbb{T} lie in the distinguished boundary of Γ . A good understanding of rational Γ -inner functions is likely to play a part in any future solution of the finite interpolation problem for $\text{Hol}(\mathbb{D}, \Gamma)$, since such a problem has a solution if and only if it has a rational Γ -inner solution (for example, [13, Theorem 4.2]). We introduce an array $\mathcal{E}_{\nu n}$ of classes of rational Γ -inner functions that are closely related both to n -extremal maps and to the conditions \mathcal{C}_ν . We say that a function $f = (s, p)$ is in $\mathcal{E}_{\nu n}$ if $f \in \text{Hol}(\mathbb{D}, \Gamma)$ is rational and there exists $m \in \mathcal{B}l_\nu$ such that

$$\frac{2mp - s}{2 - ms} \in \mathcal{B}l_{n-1}.$$

Here $\mathcal{B}l_n$ is the set of Blaschke products of degree at most n . We show that any function in $\mathcal{E}_{\nu n}$ either maps into the topological boundary of Γ or is n -extremal, while if the Γ -interpolation Conjecture holds, then any rational n -extremal Γ -inner function belongs to $\mathcal{E}_{n-2, n}$. We obtain numerous strict inclusions between \mathcal{E} classes, which are summarised in a table in Section 13.

Here is some terminology and notation.

The closed unit disc $\{z : |z| \leq 1\}$ will be denoted by Δ . The closure of a set S in a topological space will be written S^- . We denote by \mathbb{T} the unit circle, by H^2 the Hardy Hilbert space on \mathbb{D} and by K the Szegő kernel:

$$K_\lambda(z) = K(z, \lambda) = (1 - \bar{\lambda}z)^{-1}, \quad \lambda, z \in \mathbb{D}.$$

For any domain Ω and any set $E \subset \mathbb{C}^N$, we denote by $\text{Hol}(\Omega, E)$ the set of analytic functions from Ω into E . The *Schur class* \mathcal{S} is the class $\text{Hol}(\mathbb{D}, \Delta)$ of functions analytic and bounded by 1 in \mathbb{D} . For a function f on a subset of the complex plane \mathbb{C} we write

$$\bar{f}(z) = \overline{f(z)}, \quad f^\vee(z) = (f(\bar{z}))^-.$$

For $\alpha \in \mathbb{C}$ we write

$$B_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha}z}.$$

In the event that $\alpha \in \mathbb{D}$ the rational function B_α is called a *Blaschke factor*. A *Möbius function* is a function of the form cB_α for some $\alpha \in \mathbb{D}$ and $c \in \mathbb{T}$. The set of all Möbius functions is the automorphism group $\text{Aut } \mathbb{D}$ of \mathbb{D} .

We denote by $d(f)$ the degree of a rational function f of one variable – that is, the maximum of the degrees of the numerator and denominator in an expression of f as a ratio of coprime polynomials.

2. n -EXTREMAL HOLOMORPHIC MAPS

Roughly speaking, a holomorphic map h between domains is n -extremal if its restriction to any n -point set yields interpolation data that are solvable, but only just. More precisely:

Definition 2.1. *Let Ω be a domain, let $E \subset \mathbb{C}^N$, let $n \geq 1$, let $\lambda_1, \dots, \lambda_n$ be distinct points in Ω and let $z_1, \dots, z_n \in E$. We say that the interpolation data*

$$\lambda_j \mapsto z_j : \Omega \rightarrow E, \quad j = 1, \dots, n,$$

are extremally solvable if there exists a map $h \in \text{Hol}(\Omega, E)$ such that $h(\lambda_j) = z_j$ for $j = 1, \dots, n$, but, for any open neighbourhood U of the closure of Ω , there is no $f \in \text{Hol}(U, E)$ such that $f(\lambda_j) = z_j$ for $j = 1, \dots, n$.

We say further that $h \in \text{Hol}(\Omega, E)$ is n -extremal (for $\text{Hol}(\Omega, E)$) if, for all choices of n distinct points $\lambda_1, \dots, \lambda_n$ in Ω , the interpolation data

$$\lambda_j \mapsto h(\lambda_j) : \Omega \rightarrow E, \quad j = 1, \dots, n,$$

are extremally solvable.

There are no 1-extremal holomorphic maps, so we shall always suppose that $n \geq 2$.

As we mentioned in the Introduction, Pick showed that a function f is n -extremal for the Schur class $\mathcal{S} = \text{Hol}(\mathbb{D}, \Delta)$ if and only if $f \in \mathcal{B}_{n-1}$. Thus n -extremals may be regarded as an analogue of the Blaschke products of degree at most $n - 1$.

It is evident that the notion of an n -extremal holomorphic map applies very generally, but in this paper we shall be mainly concerned with n -extremals for $\text{Hol}(\mathbb{D}, \Gamma)$. We shall however point out some simple general properties of n -extremals.

Firstly, if h is an n -extremal for $\text{Hol}(\Omega, E)$ then it is also an m -extremal for all $m \geq n$. This is immediate from the definition.

Secondly we note a property of compositions of holomorphic maps: if $g \circ \alpha$ is n -extremal then so is α .

Proposition 2.2. *If E is a set in \mathbb{C}^N , Ω_1, Ω_2 are domains and $g \circ \alpha$ is n -extremal for some $\alpha \in \text{Hol}(\Omega_1, \Omega_2)$ and $g \in \text{Hol}(\Omega_2, E)$ then α is n -extremal.*

Proof. For if $\alpha \in \text{Hol}(\Omega_1, \Omega_2)$ is *not* n -extremal then there exist distinct points $\lambda_1, \dots, \lambda_n \in \Omega_1$, an open neighbourhood U of the closure of Ω_1 and an $f \in \text{Hol}(U, \Omega_2)$ such that f and α agree on $\lambda_1, \dots, \lambda_n$. Then $g \circ f \in \text{Hol}(U, E)$ and $g \circ \alpha$ and $g \circ f$ agree on $\lambda_1, \dots, \lambda_n$. This shows that $g \circ \alpha$ is not n -extremal.

Thirdly we consider the question of the holomorphic invariance of the notion of n -extremal. If g is n -extremal in $\text{Hol}(\Omega, E)$ and α is an automorphism of Ω , is it the case that $g \circ \alpha$ is n -extremal? If α extends holomorphically to a neighbourhood of the closure of Ω then the answer is yes.

Proposition 2.3. *Let $\alpha : \Omega_1 \rightarrow \Omega_2$ be a biholomorphic map of domains that extends to a biholomorphic map from an open neighbourhood U of Ω_1^- to an open neighbourhood V of Ω_2^- . If g is n -extremal in $\text{Hol}(\Omega_2, E)$ then $g \circ \alpha$ is n -extremal in $\text{Hol}(\Omega_1, E)$.*

Proof. Suppose that $g \circ \alpha$ is not n -extremal: there exist distinct points $\lambda_1, \dots, \lambda_n \in \Omega_1$, an open neighbourhood U_1 of Ω_1^- and a function $f \in \text{Hol}(U_1, E)$ such that

$$f(\lambda_j) = g \circ \alpha(\lambda_j) \quad \text{for } j = 1, \dots, n.$$

By hypothesis α extends biholomorphically to an open neighbourhood U of Ω_1^- , and therefore maps $U \cap U_1$ onto an open neighbourhood V_1 of Ω_2^- . Thus $f \circ \alpha^{-1} \in \text{Hol}(V_1, E)$, and $f \circ \alpha^{-1}$ agrees with g at the n points $\alpha(\lambda_1), \dots, \alpha(\lambda_n)$. Hence g is not n -extremal.

However, for general Ω and automorphism α there is no reason to expect α to extend even continuously to the closure of Ω , and it therefore seems unlikely that n -extremality is preserved under composition on the right with automorphisms in general.

There is a dual result to Proposition 2.3: it shows that n -extremality is better behaved with respect to composition on the left by an automorphism. Let us say that $\alpha : E_1 \rightarrow E_2$ is an isomorphism, for any pair of sets E_1, E_2 , if α is bijective and α, α^{-1} are complex-differentiable on E_1, E_2 respectively.

Proposition 2.4. *Let Ω be a domain and let $\alpha : E_1 \rightarrow E_2$ be an isomorphism. Then $g \in \text{Hol}(\Omega, E_1)$ is n -extremal if and only if $\alpha \circ g$ is n -extremal in $\text{Hol}(\Omega, E_2)$.*

Proof. Suppose that g is n -extremal but $\alpha \circ g$ is not n -extremal. Then there exist distinct points $\lambda_1, \dots, \lambda_n \in \Omega$, an open neighbourhood U of Ω^- and a function $f \in \text{Hol}(U, E_2)$ such that

$$f(\lambda_j) = \alpha \circ g(\lambda_j) \quad \text{for } j = 1, \dots, n.$$

Since α^{-1} is complex-differentiable, $\alpha^{-1} \circ f \in \text{Hol}(U, E_1)$ (the property of complex-differentiability is vacuous at any isolated point of E_2 , but this does not matter). Furthermore

$$\alpha^{-1} \circ f(\lambda_j) = g(\lambda_j) \quad \text{for } j = 1, \dots, n,$$

contrary to the hypothesis that g is n -extremal. Thus g n -extremal implies that $\alpha \circ g$ is n -extremal. On applying this result to $\alpha \circ g \in \text{Hol}(\Omega, E_2)$ and $\alpha^{-1} : E_2 \rightarrow E_1$ we obtain the converse statement.

Our fourth observation relates to the question: for which n and Ω is the identity map id_Ω n -extremal? We do not know any domain for which the identity map is not 2-extremal, and the following proposition gives a significant class of domains for which id_Ω is 2-extremal.

Proposition 2.5. *Let Ω be a bounded domain in \mathbb{C}^N such that, for every pair of distinct points $z_1, z_2 \in \Omega$ there is a rational Kobayashi extremal function for z_1, z_2 . Then id_Ω is 2-extremal.*

We defer the explanation and proof of this statement to the next subsection.

2.1. Complex geodesics and 2-extremals. The n -extremal holomorphic maps also generalise a class of functions that are important in complex geometry. Consider any domain $\Omega \subset \mathbb{C}^N$. Let δ_Ω denote the Lempert function of Ω , defined for $z_1, z_2 \in \Omega$ by

$$(2.1) \quad \delta_\Omega(z_1, z_2) \stackrel{\text{def}}{=} \inf \{ \rho(\mu_1, \mu_2) : \text{there exists } f \in \text{Hol}(\mathbb{D}, \Omega) \text{ such that} \\ f(\mu_1) = z_1, f(\mu_2) = z_2 \}$$

where ρ denotes the pseudohyperbolic distance on \mathbb{D} ,

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{w}z} \right|.$$

Let us say that $h \in \text{Hol}(\mathbb{D}, \Omega)$ is a *Kobayashi disc* in Ω if, for every $\lambda_1, \lambda_2 \in \mathbb{D}$,

$$(2.2) \quad \delta_\Omega(h(\lambda_1), h(\lambda_2)) = \rho(\lambda_1, \lambda_2)$$

(Kobayashi [22, Chapter 4, Section 6] calls h an extremal disc if equation (2.2) holds for *some* pair of distinct points $\lambda_1, \lambda_2 \in \mathbb{D}$).

Proposition 2.6. *A function $h \in \text{Hol}(\mathbb{D}, \Omega)$ is 2-extremal if and only if h is a Kobayashi disc in Ω .*

Proof. \Leftarrow Let h be a Kobayashi disc in Ω . Suppose that h is not 2-extremal: then there exist distinct $\lambda_1, \lambda_2 \in \mathbb{D}$, a real number $r > 1$ and $f \in \text{Hol}(r\mathbb{D}, \Omega)$ such that $f(\lambda_1) = h(\lambda_1)$ and $f(\lambda_2) = h(\lambda_2)$. Define

$$f_r \in \text{Hol}(\mathbb{D}, \Omega) : \lambda \mapsto f(r\lambda).$$

We have $f_r(\lambda_j/r) = f(\lambda_j) = h(\lambda_j)$, and so, since h is a Kobayashi disc,

$$\rho(\lambda_1, \lambda_2) \leq \rho(\lambda_1/r, \lambda_2/r).$$

Hence

$$|r^2 - \bar{\lambda}_2 \lambda_1| \leq r|1 - \bar{\lambda}_2 \lambda_1|.$$

On squaring and expanding we find that

$$r^2(r^2 - 1) \leq (r^2 - 1)|\lambda_1 \lambda_2|^2,$$

which is a contradiction since $r > 1$ and $\lambda_1, \lambda_2 \in \mathbb{D}$. Thus h is 2-extremal.

\Rightarrow Let h be 2-extremal. Suppose that h is not a Kobayashi disc in Ω : then there exist $\lambda_1, \lambda_2 \in \mathbb{D}$ such that

$$\delta_\Omega(h(\lambda_1), h(\lambda_2)) < \rho(\lambda_1, \lambda_2).$$

Hence there are $\mu_1, \mu_2 \in \mathbb{D}$ and $f \in \text{Hol}(\mathbb{D}, \Omega)$ such that $f(\mu_j) = h(\lambda_j)$, $j = 1, 2$ and

$$\rho(\mu_1, \mu_2) < \rho(\lambda_1, \lambda_2).$$

By composing f, h with automorphisms of \mathbb{D} we can arrange that $\lambda_1 = \mu_1 = 0$ and

$$0 < \mu_2 < \lambda_2 < 1.$$

Let $r = \lambda_2/\mu_2 > 1$. Define

$$g \in \text{Hol}(r\mathbb{D}, \Omega) : \lambda \mapsto f(\lambda/r).$$

We have $g(0) = f(0) = h(0)$ and

$$g(\lambda_2) = f(\lambda_2/r) = f(\mu_2) = h(\lambda_2).$$

This contradicts the hypothesis that h be 2-extremal. Thus h is a Kobayashi disc in Ω .

For a large class of domains, the Kobayashi discs coincide with the complex geodesics. We recall that an analytic function $h : \mathbb{D} \rightarrow \Omega$ is called a *complex geodesic of Ω* if there exists an analytic left inverse $g : \Omega \rightarrow \mathbb{D}$ of h .

There is a dual notion to the Lempert function: the *Carathéodory pseudodistance* C_Ω on Ω is given by

$$C_\Omega(z_1, z_2) = \sup\{\rho(F(z_1), F(z_2)) : F \in \text{Hol}(\Omega, \mathbb{D})\}.$$

Any function $F \in \text{Hol}(\Omega, \mathbb{D})$ for which the supremum on the right-hand side is attained is called a *Carathéodory extremal function* for z_1, z_2 and the domain Ω .

Corollary 2.7. *Let Ω be a domain for which $\delta_\Omega = C_\Omega$. The 2-extremals for $\text{Hol}(\mathbb{D}, \Omega)$ coincide with the complex geodesics of Ω .*

For in such domains the Kobayashi discs coincide with the complex geodesics [22, Corollary 4.6.2].

By Lempert's theorem [24], [22, Theorem 4.8.13] the equality $\delta_\Omega = C_\Omega$ holds for convex domains, but there are also nonconvex domains for which it holds, including the symmetrised bidisc \mathbb{G} , which is not isomorphic to any convex domain [12].

Let us return to Proposition 2.5. A *Kobayashi extremal function* for a pair of distinct points z_1, z_2 in a domain Ω is a function $h \in \text{Hol}(\mathbb{D}, \Omega)$ which is extremal for the infimum in equation (2.1); thus, for such an h , there exist $\lambda_1, \lambda_2 \in \mathbb{D}$ such that

$$(2.3) \quad h(\lambda_1) = z_1, \quad h(\lambda_2) = z_2 \quad \text{and} \quad \rho(\lambda_1, \lambda_2) = \delta_\Omega(z_1, z_2).$$

Proof of Proposition 2.5. Suppose id_Ω is not 2-extremal. Then there exist $z_1, z_2 \in \Omega$, an open neighbourhood U of the closure of Ω and $g \in \text{Hol}(U, \Omega)$ such that $g(z_j) = z_j$ for $j = 1, 2$. By hypothesis there is a rational Kobayashi extremal function $h \in \text{Hol}(\mathbb{D}, \Omega)$ for z_1, z_2 , so that equations (2.3) hold. Since h is bounded it has no pole on \mathbb{T} , and therefore h is analytic on some open neighbourhood of Δ . Hence there exists $t > 1$ such that $h(t\mathbb{D}) \subset U$. Let $h_t(\lambda) = h(t\lambda)$ for $\lambda \in \mathbb{D}$: then $g \circ h_t \in \text{Hol}(\mathbb{D}, \Omega)$ and $g \circ h_t(\lambda_j/t) = z_j$ for $j = 1, 2$. Hence

$$\begin{aligned} \delta_\Omega(z_1, z_2) &\leq \rho(\lambda_1/t, \lambda_2/t) \\ &< \rho(\lambda_1, \lambda_2) \\ &= \delta_\Omega(z_1, z_2), \end{aligned}$$

which is a contradiction. Thus id_Ω is 2-extremal. □

2.2. Universal Carathéodory sets and n -extremal maps. In this subsection we consider a question related to the “Lempert duality” between $\text{Hol}(\mathbb{D}, \Omega)$ and $\text{Hol}(\Omega, \mathbb{D})$, mentioned in the Introduction, and to solvability criteria for interpolation problems. Can we test a map $h \in \text{Hol}(\Omega_1, \Omega_2)$ for n -extremality by examining all its compositions with a suitable subset of $\text{Hol}(\Omega_2, \mathbb{D})$? By Proposition 2.2, if $F \circ h$ is n -extremal for some $F \in \text{Hol}(\Omega_2, \mathbb{D})$, then h is n -extremal. We ask whether there is a converse implication.

Definition 2.8. *We say that a subset $C \subset \text{Hol}(\Omega, \mathbb{D})$ is a universal Carathéodory set for a domain Ω if, for every pair $z_1, z_2 \in \Omega$ there is a Carathéodory extremal function for z_1, z_2 that belongs to C .*

Example 2.9. For many classical domains Ω there are small universal Carathéodory sets for Ω .

(i) If $\Omega = \mathbb{D}^d$ then the set of the d co-ordinate functions is a universal Carathéodory set for \mathbb{D}^d .

(ii) For the ball \mathbb{B}_d in \mathbb{C}^d , the projections onto the planes through the centre constitute a universal Carathéodory set.

(iii) If $\Omega = \mathbb{G}$, the symmetrised bidisc, there is a 1-parameter set $\{\Phi_\omega : \omega \in \mathbb{T}\}$ (see Definition 3.1 below) that constitutes a universal Carathéodory set for \mathbb{G} [6].

Given a domain Ω , a universal Carathéodory set C for Ω and an integer n , we may pose:

Question 2.10. *Is it true that $h \in \text{Hol}(\mathbb{D}, \Omega)$ is n -extremal if and only if $F \circ h$ is n -extremal for every $F \in C$?*

Example 2.11. The answer to Question 2.10 depends on the domain Ω and on n .

(i) It is yes for the polydisc \mathbb{D}^d . It is easy to see that $h = (h^1, \dots, h^d)$ is n -extremal if and only if some component h^j is n -extremal.

(ii) If $\Omega = \mathbb{G}$, for $n \geq 3$, the answer to this question is no while for $n = 2$ the answer is yes. In Proposition 12.1 we construct an analytic disc h in \mathbb{G} such that h is 3-extremal and yet, for all $\omega \in \mathbb{T}$, $\Phi_\omega \circ h$ is not 3-extremal.

3. THE SYMMETRISED BIDISC Γ

We began the study of the open symmetrised bidisc \mathbb{G} in [2] to [7] with the aim of solving a special case of the μ -synthesis problem of H^∞ control: see our concluding reflections in Section 14 below. The original goal has still not been attained, but significant progress has been made, and the function theory of \mathbb{G} has turned out to be of great interest to specialists in several complex variables. The present study of n -extremal functions in $\text{Hol}(\mathbb{D}, \mathbb{G})$ throws further light on interpolation problems for \mathbb{G} .

Here we summarise the relevant facts about \mathbb{G} . We repeat the definitions: the *open* and *closed symmetrised bidiscs* are defined to be the sets

$$\mathbb{G} \stackrel{\text{def}}{=} \{(z + w, zw) : |z| < 1, |w| < 1\} \quad \text{and}$$

$$\Gamma \stackrel{\text{def}}{=} \{(z + w, zw) : |z| \leq 1, |w| \leq 1\}$$

respectively. It is evident that the domain \mathbb{G} is closely related to the bidisc, but \mathbb{G} has a richer structure. Its distinguished boundary is topologically a Möbius band, and so is inhomogeneous (unlike that of the bidisc). The (equal) invariant distances $\delta_{\mathbb{G}}$ and $C_{\mathbb{G}}$ are less simple than for \mathbb{D}^2 , but they can be calculated fairly explicitly [6]. The complex geodesics can also be described explicitly [7, 28]; they are rational of degree at most 2, and so, by Proposition 2.5, $\text{id}_{\mathbb{G}}$ is 2-extremal.

The distinguished boundary of \mathbb{G} (or Γ) will be denoted by $b\Gamma$. Thus $b\Gamma$ is the Šilov boundary of the algebra of continuous functions on Γ that are analytic in \mathbb{G} . It is the symmetrisation of the 2-torus:

$$b\Gamma = \{(z + w, zw) : |z| = |w| = 1\}.$$

Certain simple rational functions play a central role in the study of Γ .

Definition 3.1. The function Φ is defined for $(z, s, p) \in \mathbb{C}^3$ such that $zs \neq 2$ by

$$\Phi(z, s, p) = \frac{2zp - s}{2 - zs}.$$

We shall write $\Phi_z(s, p)$ as a synonym for $\Phi(z, s, p)$.

In particular, Φ is defined and analytic on $\mathbb{D} \times \Gamma$ (since $|s| \leq 2$ when $(s, p) \in \Gamma$). See [3] for an account of how Φ arises from operator-theoretic considerations.

It will be useful to have criteria for a point of \mathbb{C}^2 to belong to Γ , $b\Gamma$ or the topological boundary $\partial\Gamma$.

Proposition 3.2. Let $(s, p) \in \mathbb{C}^2$. Then

- (1) $(s, p) \in \mathbb{G} \Leftrightarrow |s - \bar{s}p| < 1 - |p|^2$;
- (2) $(s, p) \in \Gamma$
 $\Leftrightarrow |s| \leq 2$ and $|s - \bar{s}p| \leq 1 - |p|^2$
 $\Leftrightarrow |s| \leq 2$ and, for all ω in a dense subset of \mathbb{T} , $|\Phi(\omega, s, p)| \leq 1$;
- (3) $(s, p) \in b\Gamma \Leftrightarrow |s| \leq 2$, $|p| = 1$ and $s = \bar{s}p$;
- (4) $(s, p) \in \partial\Gamma$
 $\Leftrightarrow |s| \leq 2$ and $|s - \bar{s}p| = 1 - |p|^2$
 \Leftrightarrow there exist $z \in \mathbb{T}$ and $w \in \Delta$ such that $s = z + w$, $p = zw$.

Furthermore, for $\omega \in \mathbb{T}$ and $(s, p) \in \Gamma$,

$$|\Phi_\omega(s, p)| = 1 \quad \text{if and only if} \quad \omega(s - \bar{s}p) = 1 - |p|^2.$$

Proof. These statements are mainly contained in [6, Theorem 2.1 and Corollary 2.2] and [7, Introduction], *inter alia*, but statement (2) is a slight refinement. The introduction of the dense subset of $\omega \in \mathbb{T}$ is occasioned by the fact that Φ_ω is defined everywhere on Γ except at the point $(2\bar{\omega}, \bar{\omega}^2)$.

To prove the second equivalence in (2), the fact that $(s, p) \in \Gamma$ implies that $|s| \leq 2$ and $|\Phi_\omega(s, p)| \leq 1$ for all but at most one $\omega \in \mathbb{T}$ is in [6, Theorem 2.1 and Corollary 2.2]. Conversely, if $|s| \leq 2$ and $|\Phi_\omega(s, p)| \leq 1$ for all ω in a dense subset of \mathbb{T} , then, by continuity, for all $\omega \in \mathbb{T}$, we have

$$|2\omega p - s|^2 \leq |2 - \omega s|^2.$$

On expanding this inequality we find that

$$4|p|^2 - 4\operatorname{Re}(\omega \bar{s}p) + |s|^2 \leq 4 - 4\operatorname{Re}(\omega s) + |s|^2.$$

Thus, for all $\omega \in \mathbb{T}$,

$$\operatorname{Re}(\omega(s - \bar{s}p)) \leq 1 - |p|^2$$

and therefore

$$|s - \bar{s}p| \leq 1 - |p|^2.$$

It follows by the first equivalence in (2) that $(s, p) \in \Gamma$.

The same calculation shows that, for fixed $\omega \in \mathbb{T}$ and $(s, p) \in \Gamma$,

$$|\Phi_\omega(s, p)| = 1 \Leftrightarrow \operatorname{Re}(\omega(s - \bar{s}p)) = 1 - |p|^2,$$

and since, by part (2), $|s - \bar{s}p| \leq 1 - |p|^2$, the last equation is true if and only if $\omega(s - \bar{s}p) = 1 - |p|^2$.

The variety

$$\mathcal{V} \stackrel{\text{def}}{=} \{(2z, z^2) : z \in \mathbb{C}\}$$

plays a special role in the study of Γ . For one thing, $\mathcal{V} \cap \mathbb{G}$ is the orbit of $\{(0, 0)\}$ under the automorphism group of \mathbb{G} . We call \mathcal{V} the *royal variety*.

4. INTERPOLATION IN $\text{Hol}(\mathbb{D}, \Gamma)$ AND A CONJECTURE

The (finite) interpolation problem for $\text{Hol}(\mathbb{D}, \Gamma)$ is the following:

Given n distinct points $\lambda_1, \dots, \lambda_n$ in the open unit disc \mathbb{D} and n points z_1, \dots, z_n in Γ , find if possible an analytic function

$$(4.1) \quad h : \mathbb{D} \rightarrow \Gamma \text{ such that } h(\lambda_j) = z_j \text{ for } j = 1, \dots, n.$$

If Γ is replaced by the closed unit disc Δ then we obtain the classical Nevanlinna-Pick problem [29], for which there is an extensive theory that furnishes among many other things a simple criterion for the existence of a solution h and an elegant parametrisation of all solutions when they exist (see for example [32, 8, 1]). The classical results extend with appropriate modifications to a narrow class of other target sets, for example to the closed unit ball of the space of $k \times k$ matrices, or more generally, to closures of Cartan domains. These extensions have applications in electrical engineering. It would be useful for engineers if we could solve the finite interpolation problem for certain further sets, and a test case that has attracted much interest is the above problem of interpolation from \mathbb{D} into Γ .

There is a satisfactory analytic theory of the problem (4.1) in the case that the number of interpolation points n is 2 (a summary and references can be found in [33]), but we are still far from understanding the problem for a general $n \in \mathbb{N}$. Here we introduce a sequence of necessary conditions for the solvability of an n -point Γ -interpolation problem and put forward a conjecture about sufficiency. In Section 12 we prove that these conditions are of strictly increasing strength.

Consideration of some examples, including the case $n = 2$, led us to the following:

Conjecture 4.1. The Γ -interpolation conjecture. *The Γ -interpolation data*

$$\lambda_j \mapsto (s_j, p_j), \quad 1 \leq j \leq n,$$

are solvable if and only if, for every Blaschke product v of degree at most $n - 2$, the data

$$\lambda_j \mapsto \frac{2v(\lambda_j)p_j - s_j}{2 - v(\lambda_j)s_j}, \quad 1 \leq j \leq n,$$

are solvable for the classical Nevanlinna-Pick problem.

Here we say that

$$(4.2) \quad \lambda_j \mapsto z_j, \quad 1 \leq j \leq n,$$

are Γ -interpolation data if $\lambda_1, \dots, \lambda_n$ are distinct points in \mathbb{D} and $z_1, \dots, z_n \in \Gamma$. The data are *solvable* if there exists an analytic function $f : \mathbb{D} \rightarrow \Gamma$ such that $f(\lambda_j) = z_j$ for $j = 1, \dots, n$; any such function is said to be a *solution* of the Γ -interpolation problem (4.1) with data (4.2). Observe that Pick's Theorem gives us an easily-checked criterion for the solvability of a Nevanlinna-Pick problem (see Proposition 4.5 below).

Conjecture 4.1 is true in the case $n = 2$ [6]; see also [33]. We have no evidence for $n \geq 3$ and we are open minded as to whether or not it is likely to be true for all n .

We shall formalise the condition which appears in Conjecture 4.1 and which plays an important role in the paper.

Definition 4.2. *Corresponding to interpolation data*

$$(4.3) \quad \lambda = (\lambda_1, \dots, \lambda_n), \quad z = (z_1, \dots, z_n),$$

where $\lambda_1, \dots, \lambda_n$ are distinct points in \mathbb{D} and $z_j = (s_j, p_j) \in \mathbb{G}$ for $j = 1, \dots, n$, we introduce:

Condition $\mathcal{C}_\nu(\lambda, z)$

For every Blaschke product v of degree at most ν , the Nevanlinna-Pick data

$$(4.4) \quad \lambda_j \mapsto \frac{2v(\lambda_j)p_j - s_j}{2 - v(\lambda_j)s_j}, \quad j = 1, \dots, n,$$

are solvable.

Thus Conjecture 4.1 can be stated: *Condition \mathcal{C}_{n-2} is necessary and sufficient for the solvability of an n -point Γ -interpolation problem.*

The conditions \mathcal{C}_ν are all necessary for the solvability of a Γ -interpolation problem.

Theorem 4.3. *Let $\lambda_1, \dots, \lambda_n$ be distinct points in \mathbb{D} and let $z_j \in \mathbb{G}$ for $j = 1, 2, \dots, n$. If there exists an analytic function $h : \mathbb{D} \rightarrow \Gamma$ such that $h(\lambda_j) = z_j$ for $j = 1, 2, \dots, n$ then, for any function v in the Schur class, the Nevanlinna-Pick data (4.4) are solvable. In particular, the condition $\mathcal{C}_\nu(\lambda, z)$ holds for every non-negative integer ν .*

Proof. Suppose that the analytic function h exists as described. Choose any function v in the Schur class. Then, by Proposition 3.2, for all $\lambda \in \mathbb{D}$,

$$|\Phi(v(\lambda), h(\lambda))| \leq 1.$$

The function $g = \Phi \circ (v, h)$ is analytic and bounded by 1 in \mathbb{D} , and satisfies the interpolation conditions (4.4). Hence the Nevanlinna-Pick data (4.4) are indeed solvable. In particular, the conclusion holds if v is a Blaschke product, and so condition $\mathcal{C}_\nu(\lambda, z)$ holds for every non-negative integer ν .

There is a special case in which Condition \mathcal{C}_0 is sufficient as well as necessary.

Theorem 4.4. *Let $\lambda_1, \dots, \lambda_n$ be distinct points in \mathbb{D} and let $z_j = (s_j, p_j) \in \mathbb{G}$, $1 \leq j \leq n$. If condition $\mathcal{C}_0(\lambda, z)$ holds and the Nevanlinna-Pick problem with data $\lambda_j \mapsto p_j$ is extremally solvable then*

$$\lambda_j \mapsto z_j, \quad 1 \leq j \leq n,$$

are solvable Γ -interpolation data.

This result is [6, Theorem 5.2].

Condition \mathcal{C}_0 does not suffice for general 3-point interpolation problems, as will follow from Theorem 12.4.

Pick's Theorem enables us to recast the necessary condition \mathcal{C}_ν in Theorem 4.3 as the positivity of a pencil of matrices.

Proposition 4.5. *If*

$$\lambda_j \mapsto z_j = (s_j, p_j), \quad 1 \leq j \leq n,$$

are interpolation data for Γ then condition $\mathcal{C}_\nu(\lambda_1, \dots, \lambda_n, z_1, \dots, z_n)$ holds if and only if, for every Blaschke product v of degree at most ν ,

$$(4.5) \quad \left[\frac{1 - v(\lambda_i)p_i\bar{p}_j\bar{v}(\lambda_j) - \frac{1}{2}v(\lambda_i)(s_i - p_i\bar{s}_j) - \frac{1}{2}(\bar{s}_j - \bar{p}_js_i)\bar{v}(\lambda_j) - \frac{1}{4}(1 - v(\lambda_i)\bar{v}(\lambda_j))s_i\bar{s}_j}{1 - \lambda_i\bar{\lambda}_j} \right]_{i,j=1}^n \geq 0.$$

Proof. By Pick's Theorem [1, Theorem 1.3], for any function v in the Schur class, the Nevanlinna-Pick data (4.4) are solvable if and only if

$$(4.6) \quad \left[\frac{1 - \bar{w}_i w_j}{1 - \bar{\lambda}_i \lambda_j} \right]_{i,j=1}^n \geq 0,$$

where

$$w_j = \frac{2v(\lambda_j)p_j - s_j}{2 - v(\lambda_j)s_j}.$$

On conjugating the inequality (4.6) by

$$\text{diag}\{2 - v(\lambda_1)s_1, \dots, 2 - v(\lambda_n)s_n\}$$

we deduce that the inequality (4.6) holds if and only if

$$\left[\frac{(2 - v(\lambda_i)s_i)(2 - \bar{v}(\lambda_j)\bar{s}_j) - (2v(\lambda_i)p_i - s_i)(2\bar{v}(\lambda_j)\bar{p}_j - \bar{s}_j)}{1 - \lambda_i\bar{\lambda}_j} \right]_{i,j=1}^n \geq 0,$$

which simplifies to the given condition (4.5). Hence condition $\mathcal{C}_\nu(\lambda, z)$ holds if and only if, for every Blaschke product v of degree at most ν , the inequality (4.5) holds.

An analogue of condition \mathcal{C}_0 for interpolation into the symmetrised polydisc in \mathbb{C}^N , for $N \geq 2$, was found by D. Ogle [27, Corollary 5.2.2]. However, when $N \geq 3$, this necessary condition is insufficient even for two-point interpolation problems [11, Observation 1.3].

5. EXTREMALITY IN CONDITION \mathcal{C}_ν

To prove that condition \mathcal{C}_ν suffices for the solvability of an n -point Nevanlinna-Pick problem for Γ it is enough to prove it in the case that \mathcal{C}_ν holds *extremally*. Let us make this notion precise.

Recall that Γ -interpolation data $\lambda_j \mapsto z_j$, $1 \leq j \leq n$, are defined to satisfy condition \mathcal{C}_ν if, for every Blaschke product $v \in \mathcal{B}l_\nu$ of degree at most ν , the data

$$(5.1) \quad \lambda_j \mapsto \Phi(v(\lambda_j), z_j), \quad 1 \leq j \leq n,$$

are solvable for the classical Nevanlinna-Pick problem. If, in addition, there exists $m \in \mathcal{B}l_\nu$ such that the data

$$\lambda_j \mapsto \Phi(m(\lambda_j), z_j), \quad 1 \leq j \leq n,$$

are *extremally* solvable Nevanlinna-Pick data, then we shall say that the data $\lambda_j \mapsto z_j$, $1 \leq j \leq n$, *satisfy \mathcal{C}_ν extremally*, or the condition $\mathcal{C}_\nu(\lambda, z)$ *holds extremally*. Here $\lambda = (\lambda_1, \dots, \lambda_n)$ and $z = (z_1, \dots, z_n)$.

It is well known (e.g. [1]) that Pick's criterion for the solvability of a classical Nevanlinna-Pick problem is expressible by an operator norm inequality; hence condition \mathcal{C}_ν can be expressed this way. Let

$$(5.2) \quad \mathcal{M} = \text{span} \{K_{\lambda_1}, \dots, K_{\lambda_n}\} \subset H^2,$$

where K is the Szegő kernel. Consider Γ -interpolation data $\lambda_j \mapsto z_j$, $1 \leq j \leq n$, and introduce, for any function v in the Schur class, the operator $X(v)$ on \mathcal{M} given by

$$(5.3) \quad X(v)K_{\lambda_j} = \overline{\Phi(v(\lambda_j), z_j)}K_{\lambda_j}, \quad 1 \leq j \leq n.$$

Pick's Theorem, as reformulated by Sarason [31], asserts that the Nevanlinna-Pick data

$$(5.4) \quad \lambda_j \mapsto \Phi(v(\lambda_j), z_j), \quad 1 \leq j \leq n,$$

are solvable if and only if the operator $X(v)$ is a contraction. Furthermore, the Nevanlinna-Pick data (5.4) are extremally solvable if and only if $\|X(v)\| = 1$.

Thus $\mathcal{C}_\nu(\lambda, z)$ holds if and only if

$$(5.5) \quad \sup_{v \in \mathcal{B}l_\nu} \|X(v)\| \leq 1.$$

Proposition 5.1. *For any Γ -interpolation data $\lambda_j \mapsto z_j$, $1 \leq j \leq n$, and $\nu \geq 0$, the following conditions are equivalent.*

- (i) $\mathcal{C}_\nu(\lambda, z)$ holds extremally;
- (ii) $\sup_{v \in \mathcal{B}l_\nu} \|X(v)\| = 1$;
- (iii) $\mathcal{C}_\nu(\lambda, z)$ holds and there exist $m \in \mathcal{B}l_\nu$ and $q \in \mathcal{B}l_{n-1}$ such that

$$(5.6) \quad \Phi(m(\lambda_j), z_j) = q(\lambda_j), \quad j = 1, \dots, n,$$

Moreover, when condition (iii) is satisfied for some $m \in \mathcal{B}l_\nu$, there is a unique $q \in \mathcal{B}l_{n-1}$ such that equations (5.6) hold. If, furthermore, the Γ -interpolation data $\lambda_j \mapsto z_j$, $1 \leq j \leq n$, are solvable by an analytic function $h = (s, p) : \mathbb{D} \rightarrow \Gamma$, then

$$(5.7) \quad \frac{2mp - s}{2 - ms} = q.$$

Proof. (i) \Rightarrow (ii). Assume (i): then the inequality (5.5) holds. Furthermore there exists $m \in \mathcal{B}l_\nu$ such that

$$(5.8) \quad \lambda_j \mapsto \Phi(m(\lambda_j), z_j), \quad 1 \leq j \leq n,$$

are extremally solvable Nevanlinna-Pick data, which implies that

$$\sup_{v \in \mathcal{B}l_\nu} \|X(v)\| = 1.$$

Hence (ii) holds.

(ii) \Rightarrow (iii). Suppose (ii). Take a maximizing sequence $(v_k) \in \mathcal{B}l_\nu$ for this supremum: it has a locally uniformly convergent subsequence, whose limit is a Blaschke product m of degree $d(m)$ at most ν . We have $\|X(m)\| = 1$, so that the Nevanlinna-Pick data

$$(5.9) \quad \lambda_j \mapsto \Phi(m(\lambda_j), z_j), \quad 1 \leq j \leq n,$$

are extremally solvable. Hence there is a unique interpolating function, q say, in the Schur class, and $q \in \mathcal{B}l_{n-1}$ [32]. Thus (iii) holds.

(iii) \Rightarrow (i). Let (iii) hold. Since the Nevanlinna-Pick data

$$(5.10) \quad \lambda_j \mapsto \Phi(m(\lambda_j), z_j), \quad 1 \leq j \leq n,$$

are solvable by a Blaschke product of degree $\leq n - 1$, they are extremally solvable [1, Lemma 6.19 and Theorem 1.3]. Thus (i) holds.

Hence (i), (ii) and (iii) are equivalent.

If $m \in \mathcal{B}l_\nu$ is such that (5.6) holds for some $q \in \mathcal{B}l_{n-1}$ then the Nevanlinna-Pick problem with data (5.10), being extremally solvable, has a unique solution in the Schur class. The solution is necessarily q [1, Theorem 6.4].

Finally, if $h = (s, p) : \mathbb{D} \rightarrow \Gamma$ is a solution of the Γ -interpolation problem $\lambda_j \mapsto z_j$, $1 \leq j \leq n$, then $\Phi \circ (m, h) \in \mathcal{S}$ solves the problem (5.10), and so, by uniqueness, equals q . That is

$$\frac{2mp - s}{2 - ms} = q.$$

Suppose that \mathcal{C}_ν holds for n -point interpolation data $\lambda_j \mapsto (s_j, p_j)$, $1 \leq j \leq n$. On replacing λ_j by $r\lambda_j$ for a suitable $r \in (0, 1]$ we can arrange that \mathcal{C}_ν holds extremally for the modified data. If condition \mathcal{C}_ν suffices for solvability of an n -point interpolation problem for Γ in the extremal case then there exists an analytic function $g : \mathbb{D} \rightarrow \Gamma$ such that $g(r\lambda_j) = z_j$, $1 \leq j \leq n$. Then the function $h(\lambda) = g(r\lambda)$ solves the original problem. This justifies the claim at the beginning of this section that it suffices to prove sufficiency in Conjecture 4.1 for the case that \mathcal{C}_ν holds extremally.

We shall say that any Blaschke product m with the properties described in Proposition 5.1(iii) is an *auxiliary extremal* for the condition $\mathcal{C}_\nu(\lambda, z)$.

Let us consider the degrees of auxiliary extremals m associated with data $\lambda_j \mapsto (s_j, p_j)$, $1 \leq j \leq 3$, that satisfy \mathcal{C}_1 extremally. It is far from the case that m is uniquely determined, or even that the degree $d(m)$ is unique for a particular set of data.

Examples 5.2. Let $\lambda_1, \lambda_2, \lambda_3$ be any three distinct points in \mathbb{D} and let $0 < r < 1$. In each of the following examples h is an analytic function from \mathbb{D} to \mathbb{G} and the data $\lambda_j \mapsto h(\lambda_j)$, $1 \leq j \leq 3$, satisfy \mathcal{C}_1 extremally.

(1) Let $h(\lambda) = (2r\lambda, \lambda^2)$. Every degree 0 inner function $m \in \mathbb{T}$ is an auxiliary extremal for \mathcal{C}_1 ; there is no auxiliary extremal of degree 1.

(2) Let $h(\lambda) = (r(1 + \lambda), \lambda)$. Every $m \in \mathcal{B}l_1$ is an auxiliary extremal for \mathcal{C}_1 . The corresponding q has degree $d(m) + 1$.

(3) Let

$$h(\lambda) = \left(2(1 - r) \frac{\lambda^2}{1 + r\lambda^3}, \frac{\lambda(\lambda^3 + r)}{1 + r\lambda^3} \right), \quad \lambda \in \mathbb{D}.$$

The function $m(\lambda) = -\lambda$ is an auxiliary extremal for \mathcal{C}_1 ; there is no auxiliary extremal of degree 0. Here $q(\lambda) = -\lambda^2$. See Proposition 12.1 for a more general example and Theorem 12.4 for applications.

(4) Let f be a Blaschke product of degree 1 or 2 and let $h = (2f, f^2)$. Every $m \in \mathcal{B}l_1$ is an auxiliary extremal and, for every m , we have $q = -f$.

6. Γ -INNER FUNCTIONS

Definition 6.1. A Γ -inner function is an analytic function $h : \mathbb{D} \rightarrow \Gamma$ such that the radial limit

$$(6.1) \quad \lim_{r \rightarrow 1-} h(r\lambda) \in b\Gamma$$

for almost all $\lambda \in \mathbb{T}$.

By Fatou's Theorem, the radial limit (6.1) exists for almost all $\lambda \in \mathbb{T}$ with respect to Lebesgue measure. Observe that, in view of Proposition 3.2(3), if $h = (h_1, h_2)$ is a Γ -inner function, then h_2 is an inner function on \mathbb{D} in the conventional sense.

The easiest way to construct a Γ -inner function is to symmetrise a pair of inner functions.

Example 6.2. Let φ and ψ be inner functions on \mathbb{D} . Then

$$(6.2) \quad h = (\varphi + \psi, \varphi\psi)$$

is Γ -inner. In particular, $h = (2\varphi, \varphi^2)$ is Γ -inner; this example has the property that $h(\mathbb{D})$ lies in the royal variety \mathcal{V} .

The analysis of the Γ -inner function (6.2) reduces to the study of pairs of conventional inner functions.

Proposition 6.3. Let $h = (s, p)$ be a Γ -inner function. Then h is of the form $(\varphi + \psi, \varphi\psi)$ for some pair φ, ψ of inner functions on \mathbb{D} if and only if $s^2 - 4p$ has an analytic square root on \mathbb{D} .

Proof. Necessity is trivial, since $s^2 - 4p = (\varphi - \psi)^2$. Conversely, if $s^2 - 4p = g^2$ where g is analytic on \mathbb{D} , then we may take $\varphi = \frac{1}{2}(s - g)$ and $\psi = \frac{1}{2}(s + g)$; we find that φ, ψ are inner and $h = (\varphi + \psi, \varphi\psi)$.

Example 6.4. Let $|\beta| < 1$. The function

$$(6.3) \quad h(\lambda) = (\beta\lambda + \bar{\beta}, \lambda)$$

is Γ -inner and is not the symmetrisation of a pair of inner functions.

Proof. Write $h = (s, p)$. It is easy to see from Proposition 3.2 that h is Γ -inner. We claim that $s^2 - 4p$ does not have an analytic square root on \mathbb{D} ; to prove this it suffices to show that $s^2 - 4p$ has a *simple* zero in \mathbb{D} . Now

$$s^2 - 4p = (\beta\lambda + \bar{\beta})^2 - 4\lambda = \beta^2\lambda^2 + 2(|\beta|^2 - 2)\lambda + (\bar{\beta})^2.$$

If $\beta = 0$ then $s^2 - 4p$ has a simple zero at 0. Otherwise, $s^2 - 4p$ is a quadratic with discriminant

$$4[(|\beta|^2 - 2)^2 - |\beta|^4] = 16(1 - |\beta|^2)$$

which is nonzero for $|\beta| < 1$, and so $s^2 - 4p$ has two distinct zeros. The sum and product of these zeros satisfy

$$|\text{sum}| = 2 \frac{2 - |\beta|^2}{|\beta|^2} > 2, \quad |\text{product}| = \left| \frac{\bar{\beta}^2}{\beta^2} \right| = 1.$$

The former inequality implies that the zeros of $s^2 - 4p$ do not both lie on \mathbb{T} and the latter then shows that one lies in \mathbb{D} and one in $\mathbb{C} \setminus \Delta$. Consequently $s^2 - 4p$ has a simple zero in \mathbb{D} , and so h is not the symmetrisation of a pair of inner functions.

The function h given by equation (6.3) is not only Γ -inner – it is a *complex geodesic* of \mathbb{G} . In the case of the function h from equation (6.3) the simplest left inverse is of course the projection $(s, p) \mapsto p$. The domain \mathbb{G} also has complex geodesics of degree 2 [7, Theorem 0.2]: these afford a further class of interesting Γ -inner functions.

6.1. New Γ -inner functions from old. There are three easy ways to construct new Γ -inner functions from a given Γ -inner function.

Proposition 6.5. *Let $h = (s, p)$ be a Γ -inner function.*

- (i) *For any inner function φ , $h \circ \varphi$ is Γ -inner.*
- (ii) *For any inner function v , (vs, v^2p) is Γ -inner.*
- (iii) *If*

$$(6.4) \quad 0 \leq r \leq \min \left\{ \frac{2}{\|s\|_\infty}, \inf_{\mathbb{D}} \frac{1 - |p|^2}{|s - \bar{s}p|} \right\}$$

then (rs, p) is Γ -inner. In particular, (rs, p) is Γ -inner whenever $0 \leq r \leq 1$.

All three statements follow easily from Proposition 3.2. The minimum on the right hand side of inequality (6.4) can be greater than 1: for $h(\lambda) = (\beta\lambda + \bar{\beta}, \lambda)$ in Example 6.4, the minimum is $\frac{1}{|\beta|}$.

Also under the heading of “new from old” comes the intriguing fact that the Γ -inner functions have a non-obvious semigroup structure; see [25, Corollary 16(1a)] and [13, Theorem 3.4].

Proposition 6.6. *If (s, p) and (t, q) are Γ -inner functions then so is $(\frac{1}{2}st, pq)$.*

Proof. Clearly $|\frac{1}{2}st| \leq 2$ on \mathbb{D} , pq is inner and

$$\frac{1}{2}st = \frac{1}{2}(\bar{s}p)(\bar{t}q) = \left(\frac{1}{2}\bar{s}\bar{t}\right)pq$$

on \mathbb{T} .

Note that Proposition 6.5(ii) is a special case of Proposition 6.6, since $(2v, v^2)$ is obviously Γ -inner for any inner function v . The constant function $(2, 1)$ is an identity for the semigroup of Γ -inner functions. The only idempotents of the semigroup are $(2, 1)$ and $(0, 1)$. The only elements that have inverses in the semigroup are the constant functions of the form $(2\omega, \omega^2)$ for some $\omega \in \mathbb{T}$. The semigroup structure will not play a role in this paper.

6.2. Rational Γ -inner functions. The following example shows that *not every pole of p need be a pole of s* for a rational Γ -inner function (s, p) .

Example 6.7. Let $a \in \mathbb{D} \setminus \{0\}$ and let

$$h(\lambda) = (s(\lambda), p(\lambda)) = \left(\frac{c\lambda}{1 - \bar{a}\lambda}, \frac{\lambda(\lambda - a)}{1 - \bar{a}\lambda} \right)$$

for some $c \in \mathbb{R}$ such that $|c| \leq 2(1 - |a|)$. It is easy to check that h is Γ -inner, and whereas p has a pole at infinity, $s(\infty) = -\frac{c}{\bar{a}} \neq \infty$. Similarly, if $b \in \mathbb{D} \setminus \{0\}$ and $(s^b, p^b) = h \circ B_b$, then (s^b, p^b) is Γ -inner, p^b has a pole at $B_{-b}(\infty) = \frac{1}{\bar{b}}$ and $s^b(\frac{1}{\bar{b}}) = -\frac{c}{\bar{a}} \neq \infty$. \square

In Example 6.7, the pole $\frac{1}{b}$ of p^b that is not a pole of s^b corresponds to the common zero b of s^b and p^b . This phenomenon is general.

Proposition 6.8. *Let (s, p) be a rational Γ -inner function.*

(i) *If $a \in \mathbb{C} \cup \{\infty\}$ is a pole of s , of multiplicity $k \geq 1$, then a is a pole of p of multiplicity at least k .*

(ii) *If $a \in \mathbb{C} \cup \{\infty\}$ is a pole of p , of multiplicity $k \geq 0$, and $\frac{1}{a}$ is a zero of s of multiplicity $\ell \geq 0$, then a is a pole of s of multiplicity at least $k - \ell$.*

Proof. (i) The equations

$$s(\lambda) = \overline{s(\bar{\lambda})}p(\lambda) = s^\vee(\bar{\lambda})p(\lambda) = s^\vee\left(\frac{1}{\lambda}\right)p(\lambda)$$

hold for all $\lambda \in \mathbb{T}$. Since the first and last terms are rational functions,

$$(6.5) \quad s(\lambda) = s^\vee\left(\frac{1}{\lambda}\right)p(\lambda) \quad \text{for all } \lambda \in \mathbb{C}.$$

Hence, if $a \in \mathbb{C}$,

$$(6.6) \quad (\lambda - a)^{k-1}s(\lambda) = s^\vee\left(\frac{1}{\lambda}\right)(\lambda - a)^{k-1}p(\lambda) \quad \text{for all } \lambda \in \mathbb{C}.$$

Since a is a pole of s , $|a| > 1$, hence $\left|\frac{1}{a}\right| < 1$, and so s^\vee is analytic at $\frac{1}{a}$. On letting $\lambda \rightarrow a$ in equation 6.6 we find that $(\lambda - a)^{k-1}p(\lambda) \rightarrow \infty$ as $\lambda \rightarrow a$. Thus a is a pole of p of multiplicity at least k .

Now suppose that ∞ is a pole of s of multiplicity $k \geq 1$. Then 0 is a pole of $s\left(\frac{1}{\lambda}\right)$ of multiplicity k , so that $\lambda^{k-1}s\left(\frac{1}{\lambda}\right) \rightarrow \infty$ as $\lambda \rightarrow 0$. From the relation (6.5) we have

$$\lambda^{k-1}s\left(\frac{1}{\lambda}\right) = s^\vee(\lambda)\lambda^{k-1}p\left(\frac{1}{\lambda}\right).$$

Since s^\vee is analytic at 0, we have $\lim_{\lambda \rightarrow 0} s^\vee(\lambda) = s^\vee(0)$. Therefore

$$\lim_{\lambda \rightarrow 0} \lambda^{k-1}p\left(\frac{1}{\lambda}\right) = \infty,$$

which is to say that p has a pole of multiplicity at least k at ∞ .

(ii) Again the statement follows on letting $\lambda \rightarrow a$ in equation (6.5), since $s^\vee\left(\frac{1}{\lambda}\right)$ has a zero of multiplicity ℓ at a .

Now suppose that ∞ is a pole of p of multiplicity $k \geq 1$ and 0 is a zero of s of multiplicity $\ell \geq 0$; then ∞ is a pole of s of multiplicity $k - \ell$. From the relation (6.5), for all $\lambda \in \mathbb{C} \setminus \{0\}$, we have

$$s\left(\frac{1}{\lambda}\right) = s^\vee(\lambda)p\left(\frac{1}{\lambda}\right),$$

and so

$$(6.7) \quad \frac{\lambda^{k-1}}{\lambda^\ell}s\left(\frac{1}{\lambda}\right) = \frac{s^\vee(\lambda)}{\lambda^\ell}\lambda^{k-1}p\left(\frac{1}{\lambda}\right).$$

Since s^\vee is analytic at 0 and has a zero at 0 of multiplicity $\ell \geq 0$, we have

$$\lim_{\lambda \rightarrow 0} \frac{s^\vee(\lambda)}{\lambda^\ell} = c \neq 0 \text{ where } c \in \mathbb{C}.$$

By assumption, $p(\lambda)$ has a pole of multiplicity k at ∞ , and so $\lim_{\lambda \rightarrow 0} \lambda^{k-1}p\left(\frac{1}{\lambda}\right) = \infty$. Therefore, by equation 6.7,

$$\lim_{\lambda \rightarrow 0} \lambda^{k-\ell-1}s\left(\frac{1}{\lambda}\right) = \infty.$$

which is to say that $s\left(\frac{1}{\lambda}\right)$ has a pole of multiplicity $k - \ell$ at 0. Thus $s(\lambda)$ has a pole of multiplicity $k - \ell$ at ∞ .

Remark 6.9. In Proposition 6.8 we allow the possibility that $\ell > k$, in which case a is a zero of s of multiplicity $\ell - k$. In Example 6.4 we consider the rational Γ -inner function $h(\lambda) = (\beta\lambda + \bar{\beta}, \lambda)$. The function $s(\lambda) = \beta\lambda + \bar{\beta}$ has a zero of multiplicity $\ell = 1$ at $\lambda = -\frac{\bar{\beta}}{\beta}$. The function $p(\lambda) = \lambda$ has a pole of multiplicity $k = 0$ at $\lambda = -\frac{\bar{\beta}}{\beta}$. Thus we have $\ell > k$.

Corollary 6.10. *If (s, p) is a rational Γ -inner function then s and p can be written as ratios of polynomials with the same denominators. More precisely, let*

$$(6.8) \quad p(\lambda) = c \frac{\lambda^k \tilde{D}_p(\lambda)}{D_p(\lambda)}$$

where $|c| = 1$, $k \geq 0$, D_p is a polynomial of degree n such that $D_p(0) = 1$ and $\tilde{D}_p(\lambda) = \lambda^n D_p^\vee\left(\frac{1}{\lambda}\right)$. Then s is expressible in the form

$$(6.9) \quad s(\lambda) = \frac{\lambda^\ell N_s(\lambda)}{D_p(\lambda)}$$

where $0 \leq \ell \leq \frac{1}{2}(n + k) = \frac{1}{2}d(p)$, and N_s is a polynomial of degree $d(p) - 2\ell$ such that $N_s(0) \neq 0$. Moreover, if $N_s(\lambda) = \sum_{j=0}^{n+k-2\ell} b_j \lambda^j$ then

$$(6.10) \quad b_j = \bar{c} \bar{b}_{n+k-2\ell-j} \text{ for } j = 0, 1, \dots, n+k-2\ell.$$

The degree of s is at most $\max\{n + k - \ell, n\}$.

Proof. Every finite Blaschke product is expressible in the form (6.8). Since every pole of s is a pole of p , s can be written as a ratio of polynomials with denominator D_p , though not necessarily in its lowest terms (cf. Example 6.7).

Let $\ell \geq 0$ be the multiplicity with which s vanishes at 0: then s can be written in the form (6.9) for some polynomial N_s . From the relation $s(\lambda) = s^\vee\left(\frac{1}{\lambda}\right) p(\lambda)$ for all $\lambda \in \mathbb{C}$ we deduce

$$(6.11) \quad \frac{\lambda^\ell N_s(\lambda)}{D_p(\lambda)} = \frac{1}{\lambda^\ell} \frac{N_s^\vee(1/\lambda)}{D_p^\vee(1/\lambda)} c \lambda^k \frac{\lambda^n D_p^\vee(1/\lambda)}{D_p(\lambda)}.$$

Hence, for all $\lambda \in \mathbb{C}$,

$$N_s(\lambda) = c \lambda^{n+k-2\ell} N_s^\vee(1/\lambda).$$

That is, if N_s has degree d ,

$$N_s(\lambda) = \sum_{j=0}^d b_j \lambda^j = c \lambda^{n+k-2\ell} \sum_{i=0}^d \frac{\bar{b}_i}{\lambda^i}.$$

The constant term b_0 is nonzero and is the term of lowest degree, and hence $0 = n + k - 2\ell - d$ and

$$b_0 = \bar{c} \bar{b}_d = \bar{c} \bar{b}_{n+k-2\ell}.$$

Similarly, on equating coefficients of λ^j , we obtain equation (6.10). From the fact that $d \geq 0$ we conclude that $2\ell \leq n + k$.

7. THE CLASSES $\mathcal{E}_{\nu k}$

Proposition 5.1 tells us that if $h \in \text{Hol}(\mathbb{D}, \Gamma)$ and $\lambda_1, \dots, \lambda_n$ are distinct points in \mathbb{D} , then the Γ -interpolation data $\lambda_j \mapsto h(\lambda_j)$ satisfy $C_\nu(\lambda, h(\lambda))$ *extremally* if and only if there exists $m \in \mathcal{B}l_\nu$ such that $\Phi \circ (m, h) \in \mathcal{B}l_{n-1}$. This leads us to introduce the following classes of rational Γ -inner functions.

Definition 7.1. For $\nu \geq 0$, $k \geq 1$ we say that the function h is in $\mathcal{E}_{\nu k}$ if $h = (s, p) \in \text{Hol}(\mathbb{D}, \Gamma)$ is rational and there exists $m \in \mathcal{B}l_\nu$ such that

$$\frac{2mp - s}{2 - ms} \in \mathcal{B}l_{k-1}.$$

Remark 7.2. It is obvious that, for every $\nu \geq 0$,

$$\mathcal{E}_{\nu 1} \subset \mathcal{E}_{\nu 2} \subset \dots \subset \mathcal{E}_{\nu k} \subset \mathcal{E}_{\nu, k+1} \subset \dots,$$

and, for every $k \geq 1$,

$$\mathcal{E}_{0k} \subset \mathcal{E}_{1k} \subset \dots \subset \mathcal{E}_{\nu k} \subset \mathcal{E}_{\nu+1, k} \subset \dots$$

We shall see in Section 9 that there is a strong connection between the class $\mathcal{E}_{\nu k}$ and k -extremality. In fact if $h \in \mathcal{E}_{\nu k}$ and $h(\mathbb{D})$ meets \mathbb{G} then h is k -extremal, while if the Γ -interpolation Conjecture is true, then every k -extremal in $\text{Hol}(\mathbb{D}, \Gamma)$ belongs to $\mathcal{E}_{k-2, k}$ (Theorem 9.1 and Observation 9.2).

It is not obvious that the functions in $\mathcal{E}_{\nu n}$ are Γ -inner, but it is so.

Theorem 7.3. Let $h \in \text{Hol}(\mathbb{D}, \Gamma)$. If there exists an inner function m such that $\Phi \circ (m, h)$ is inner then h is Γ -inner. If furthermore $h \in \mathcal{E}_{\nu n}$ for some integers $\nu \geq 0, n \geq 1$ then h is rational of degree at most $2n - 2$.

Proof. Let $h = (s, p)$ and let $q = \Phi \circ (m, h)$. Consider first the case that $mq = 1$ identically. Then m, q are constant – say $m = \omega$, $q = \bar{\omega}$ for some $\omega \in \mathbb{T}$. The relation $\Phi(\omega, s, p) = \bar{\omega}$ tells us that p is constant and equal to $\bar{\omega}^2$, and then the fact that

$$|s - \bar{s}p| \leq 1 - |p|^2 = 0$$

shows that $\omega s = \bar{\omega} \bar{s}$, and hence ωs is a real constant. Thus in this case h is constant and equal to $(x\bar{\omega}, \bar{\omega}^2)$ for some $x \in [-2, 2]$, which implies that h is Γ -inner and rational of degree 0.

The remaining case is that mq is not identically equal to 1, and so $1 - mq \neq 0$ a.e. on \mathbb{T} . Since q is inner, for almost all $\lambda \in \mathbb{T}$ we have

$$|\Phi_{m(\lambda)}(s(\lambda), p(\lambda))| = 1,$$

and so, by Proposition 3.2,

$$(7.1) \quad m(\lambda)(s - \bar{s}p)(\lambda) = 1 - |p(\lambda)|^2.$$

On rearranging the equation $\Phi(m, s, p) = q$ we obtain

$$(7.2) \quad s = 2 \frac{mp - q}{1 - mq},$$

and so equation (7.1) becomes

$$m \left(2 \frac{mp - q}{1 - mq} - 2 \frac{\bar{m}\bar{p} - \bar{q}}{1 - \bar{m}\bar{q}} p \right) (\lambda) = 1 - |p(\lambda)|^2 \quad \text{a.e. on } \mathbb{T}.$$

Since $|mq(\lambda)| = 1$ a.e. we may multiply above and below in the second term on the left hand side to obtain

$$\begin{aligned} 1 - |p(\lambda)|^2 &= 2m \left(\frac{mp - q}{1 - mq} - \frac{q\bar{p} - m}{mq - 1} p \right) (\lambda) \\ &= \frac{2m}{1 - mq} (mp - q - (q\bar{p} - m)p) (\lambda) \\ &= -\frac{2m}{1 - mq} (\lambda) (1 - |p(\lambda)|^2) \quad \text{a.e.} \end{aligned}$$

Thus

$$(1 - |p(\lambda)|^2) \frac{1 + mq}{1 - mq} (\lambda) = 0$$

a.e. on \mathbb{T} , and so p is inner.

Since $h(\mathbb{D}) \subset \Gamma$, we have $|s(\lambda)| \leq 2$ a.e. on \mathbb{T} , and by equation (7.1) we also have

$$s(\lambda) = \overline{s(\lambda)} p(\lambda) \quad \text{a.e.}$$

and so, by Proposition 3.2(3), $h(\lambda) \in b\Gamma$ for almost all $\lambda \in \mathbb{T}$, that is, h is Γ -inner.

We prove the second statement. Suppose $h \in \mathcal{E}_{\nu n}$. There exist $m \in \mathcal{B}l_{\nu}$, $q \in \mathcal{B}l_{n-1}$ such that $\Phi \circ (m, h) = q$. From the relation (7.2) and the fact that $|s(\lambda)| \leq 2$, we have

$$|mp - q|^2 \leq |1 - mq|^2$$

on \mathbb{T} . Since m, p and q are all unimodular on \mathbb{T} , on expanding we find that

$$1 - 2\operatorname{Re}(mp\bar{q}) + 1 \leq 1 - 2\operatorname{Re}(mq) + 1,$$

on \mathbb{T} , and so

$$\operatorname{Re}(mq) \leq \operatorname{Re}(mp\bar{q})$$

everywhere on \mathbb{T} . Since $mp(\lambda)$, $mp\bar{q}(\lambda) \in \mathbb{T}$ for all $\lambda \in \mathbb{T}$, it follows that if $mp\bar{q}(\lambda) = -1$ then also $mq(\lambda) = -1$. Now the finite Blaschke product mq takes on the value -1 precisely $d(mq)$ times on \mathbb{T} , and hence $mp\bar{q}$ can take the value -1 at most $d(mq)$ times. The winding number wno of $mp\bar{q}$ about 0 is thus at most $d(mq)$, that is

$$d(m) + d(p) - d(q) = \operatorname{wno}(mp\bar{q}) \leq d(m) + d(q),$$

and therefore

$$d(p) \leq 2d(q) \leq 2n - 2.$$

Proposition 7.4. *If $h \in \mathcal{E}_{\nu n}$ then, for any choice of distinct points $\lambda_1, \dots, \lambda_n \in \mathbb{D}$, the data*

$$\lambda_j \mapsto h(\lambda_j), \quad 1 \leq j \leq n,$$

satisfy $\mathcal{C}_{\nu}(\lambda, z)$ extremally where $z_j = h(\lambda_j)$, $1 \leq j \leq n$.

Proof. It follows from Proposition 5.1.

7.1. Phasar derivatives. To study the classes $\mathcal{E}_{\nu k}$ we shall need the following basic notions relating to functions on the unit circle.

Definition 7.5. *For any differentiable function $f : \mathbb{T} \rightarrow \mathbb{C} \setminus \{0\}$ the phasar derivative of f at $z = e^{i\theta} \in \mathbb{T}$ is the derivative with respect to θ of the argument of $f(e^{i\theta})$ at θ ; we denote it by $Af(z)$.*

Thus if $f(e^{i\theta}) = R(\theta)e^{ig(\theta)}$ is differentiable where $g(\theta) \in \mathbb{R}$ and $R(\theta) > 0$ then g is differentiable on $[0, 2\pi)$ and the phasar derivative of f at $z = e^{i\theta} \in \mathbb{T}$ is equal to

$$(7.3) \quad Af(e^{i\theta}) = \frac{d}{d\theta} \arg f(e^{i\theta}) = g'(\theta).$$

Clearly, for differentiable functions $\psi, \varphi : \mathbb{T} \rightarrow \mathbb{C} \setminus \{0\}$ and for any $c \in \mathbb{C} \setminus \{0\}$, we have

$$(7.4) \quad A(\psi\varphi) = A\psi + A\varphi \quad \text{and} \quad A(c\psi) = A\psi.$$

The following is a simple calculation.

Proposition 7.6. *Let $\varphi : \mathbb{T} \rightarrow \mathbb{C} \setminus \{0\}$ be a rational inner function. Then, for all $\lambda \in \mathbb{T}$,*

$$(7.5) \quad A\varphi(\lambda) = \lambda \frac{\varphi'(\lambda)}{\varphi(\lambda)}.$$

Proposition 7.7. (i) *Let*

$$B_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha}z}$$

be a Blaschke factor for $\alpha \in \mathbb{D}$. Then the phasar derivative $AB_\alpha > 0$ on \mathbb{T} .

(ii) *Let p be a rational inner function. Then the phasar derivative $Ap(\lambda) > 0$ for all $\lambda \in \mathbb{T}$.*

Proof. (i) By Proposition 7.6, for all $\alpha \in \mathbb{D}$ and $\lambda \in \mathbb{T}$, we have

$$(7.6) \quad AB_\alpha(\lambda) = \lambda \frac{B'_\alpha(\lambda)}{B_\alpha(\lambda)} = \frac{1 - |\alpha|^2}{|1 - \bar{\alpha}\lambda|^2} > 0.$$

(ii) For such p , it is well known that there exist $c \in \mathbb{T}$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{D}$ such that $p = cB_{\alpha_1} \dots B_{\alpha_n}$. Then, by Remark 7.4 and Part (i),

$$Ap = AB_{\alpha_1} + AB_{\alpha_2} + \dots + AB_{\alpha_n} > 0.$$

7.2. Cancellations and the classes $\mathcal{E}_{\nu k}$. Cancellations in the functions $\Phi \circ (v, h)$, where h is a rational Γ -inner function, are at the heart of the technical results of this paper.

Consider a rational Γ -inner function $h = (s, p)$ where

$$s(\lambda) = \frac{\lambda^\ell N_s(\lambda)}{D_p(\lambda)} \quad \text{and} \quad p(\lambda) = \frac{\lambda^k \tilde{D}_p(\lambda)}{D_p(\lambda)}$$

where $\ell \leq \frac{1}{2}d(p)$ and $d(N_s) = d(p) - 2\ell$; see Corollary 6.10.

For $\nu \geq 0$ and $v \in \mathcal{B}_\nu$, consider the function

$$\Phi \circ (v, h) = \frac{2vp - s}{2 - vs}.$$

What is the degree of the rational function $\Phi \circ (v, h)$? Since the denominator of s divides the denominator of p (see Corollary 6.10), the function $\Phi \circ (v, h)$ is a Blaschke product of degree at most $d(vp)$. If $\Phi \circ (v, h)$ has no cancellations then it has degree exactly $d(vp)$. It transpires that cancellations can only happen at special points on the unit circle.

Definition 7.8. A point $\lambda \in \Delta$ is a royal node of a Γ -inner function h if and only if $h(\lambda)$ is in the royal variety $\mathcal{V} = \{(2z, z^2) : z \in \mathbb{C}\}$.

Clearly a point $\lambda \in \Delta$ is a royal node of a Γ -inner function $h = (s, p)$ if and only if

$$(7.7) \quad s(\lambda)^2 = 4p(\lambda).$$

Definition 7.9. A Γ -inner function $h = (s, p)$ is full if $\|s\|_\infty = 2$.

Lemma 7.10. Let h be a rational Γ -inner function.

- (i) The royal nodes of $h = (s, p)$ on \mathbb{T} are precisely the points $\lambda \in \mathbb{T}$ such that $|s(\lambda)| = 2$.
- (ii) The function h has a royal node on \mathbb{T} if and only if h is full.

Proof. (i) By Remark 7.7, $\lambda \in \mathbb{T}$ is a royal node of $h = (s, p)$ if and only if $s^2(\lambda) = 4p(\lambda)$. Note that

$$|s(\lambda)| = 2 \quad \Leftrightarrow \quad s(\lambda)\overline{s(\lambda)} = 4.$$

By Proposition 3.2(3), for every $\lambda \in \mathbb{T}$, $\overline{s(\lambda)}p(\lambda) = s(\lambda)$. Hence $|s(\lambda)| = 2$ implies $s(\lambda)\overline{s(\lambda)}p(\lambda) = 4p(\lambda)$, and so $s^2(\lambda) = 4p(\lambda)$. On the other hand, by Proposition 3.2(3), for every $\lambda \in \mathbb{T}$, $|p(\lambda)| = 1$. Therefore, $s^2(\lambda) = 4p(\lambda)$ implies $|s(\lambda)|^2 = 4|p(\lambda)| = 4$.

(ii) If h has a royal node λ on \mathbb{T} then, by Part (i) h is full. Suppose $\|s\|_\infty = 2$. By the Maximum Principle, there is a $\lambda \in \mathbb{T}$ such that $|s(\lambda)| = 2$. By Part (i), $\lambda \in \mathbb{T}$ is a royal node of h .

Proposition 7.11. Let $h = (s, p)$ be a rational Γ -inner function. Suppose v is a finite Blaschke product such that

$$\Phi \circ (v, h) = \frac{2vp - s}{2 - vs}$$

has a cancellation at a point $\zeta \in \mathbb{C}$. Then h is full, $\zeta \in \mathbb{T}$, ζ is a royal node for h , $v(\zeta) = \frac{1}{2}\overline{s(\zeta)}$ and $|s(\zeta)| = 2$.

Proof. By assumption, $\Phi \circ (v, h)$ has a cancellation at ζ , and so

$$(2vp - s)(\zeta) = 0 = (2 - vs)(\zeta).$$

Thus $v(\zeta)s(\zeta) = 2$ and $2v(\zeta)s(\zeta)p(\zeta) = s^2(\zeta)$. Therefore, $s^2(\zeta) = 4p(\zeta)$, and so ζ is a royal node for h . One can also see that $2v^2(\zeta)p(\zeta) = v(\zeta)s(\zeta) = 2$. Since v is a finite Blaschke product, the equality $v^2(\zeta)p(\zeta) = 1$ implies that $\zeta \in \mathbb{T}$. By Lemma 7.10, h is full and $|s(\zeta)| = 2$. Note that $v(\zeta)s(\zeta) = 2$ and $|s(\zeta)| = 2$ imply that $v(\zeta) = \frac{1}{2}\overline{s(\zeta)}$.

Theorem 7.12. *Let $h = (s, p)$ be a nonconstant rational Γ -inner function and let v be a finite Blaschke product. Then $\Phi \circ (v, h)$ has a cancellation at ζ if and only if the following conditions are satisfied: $\zeta \in \mathbb{T}$, ζ is a royal node for h and $v(\zeta) = \frac{1}{2}\overline{s(\zeta)}$. Moreover $\Phi \circ (v, h)$ has at most one cancellation at any royal node ζ .*

Proof. Necessity follows by Proposition 7.11.

Conversely, let ζ be a royal node for h on \mathbb{T} , let $h(\zeta) = (2\bar{\omega}, \bar{\omega}^2)$ where $\omega \in \mathbb{T}$ and let $v(\zeta) = \omega$. Hence $(2vp - s)(\zeta) = 2\omega\bar{\omega}^2 - 2\bar{\omega} = 0$ and $(2 - vs)(\zeta) = 2 - \omega \cdot 2\bar{\omega} = 0$. Thus $\Phi \circ (v, h)$ has at least one cancellation at ζ .

Suppose there are 2 cancellations at ζ , so that

$$(2vp - s)'(\zeta) = 0 = (2 - vs)'(\zeta).$$

Then

$$\begin{aligned} (2vp - s)'(\zeta) &= 2v'(\zeta)p(\zeta) + 2v(\zeta)p'(\zeta) - s'(\zeta) \\ (7.8) \quad &= 2v'(\zeta)\bar{\omega}^2 + 2\omega p'(\zeta) - s'(\zeta) = 0 \end{aligned}$$

and

$$\begin{aligned} (2 - vs)'(\zeta) &= -v'(\zeta)s(\zeta) - v(\zeta)s'(\zeta) \\ (7.9) \quad &= -v'(\zeta)2\bar{\omega} - \omega s'(\zeta) = 0. \end{aligned}$$

Hence we have $v'(\zeta) = -\frac{1}{2}\omega^2 s'(\zeta)$. Then equation (7.8) can be written as

$$-2\frac{1}{2}\omega^2 s'(\zeta)\bar{\omega}^2 + 2\omega p'(\zeta) - s'(\zeta) = 0.$$

Therefore $s'(\zeta) = \omega p'(\zeta)$. By Proposition 7.6, the phasar derivatives

$$Ap(\zeta) = \zeta \frac{p'(\zeta)}{p(\zeta)} = \zeta \frac{\bar{\omega} s'(\zeta)}{\bar{\omega}^2} = \zeta \omega s'(\zeta).$$

and

$$Av(\zeta) = \zeta \frac{v'(\zeta)}{v(\zeta)} = \zeta \frac{-\frac{1}{2}\omega^2 s'(\zeta)}{\omega} = -\frac{1}{2}\zeta \omega s'(\zeta) = -\frac{1}{2}Ap(\zeta).$$

In view of Proposition 7.7, we have a contradiction since the phasar derivatives $Ap(\zeta) > 0$ and $Av(\zeta) \geq 0$. Thus the function $\Phi \circ (v, h)$ has exactly one cancellation at ζ .

Proposition 7.13. *Let $h = (s, p)$ be a rational Γ -inner function. Suppose h is not full. Then, for all $\nu \geq 0$, $h \in \mathcal{E}_{\nu, \nu+d(p)+1} \setminus \mathcal{E}_{\nu, \nu+d(p)}$.*

Proof. Since h is not full, $|s(\lambda)| < 2$ for all $\lambda \in \Delta$. By Proposition 7.11, for all $\nu \geq 0$ and all $v \in \mathcal{B}l_\nu$, $\Phi \circ (v, h)$ has no cancellations, and so has degree $d(vp)$.

Corollary 7.14. *Let h be a rational Γ -inner function of the form $(0, p)$. Then, for all $\nu \geq 0$, $h \in \mathcal{E}_{\nu, \nu+d(p)+1} \setminus \mathcal{E}_{\nu, \nu+d(p)}$.*

Proof. It follows from Proposition 7.13.

Corollary 7.15. *Let $h = (s, p)$ be a rational Γ -inner function, and let ζ_1, \dots, ζ_N be distinct royal nodes for h on \mathbb{T} . Let $h(\zeta_j) = (2\bar{\omega}_j, \bar{\omega}_j^2)$ for $j = 1, \dots, N$. If v is a finite Blaschke product such that $v(\zeta_j) = \omega_j$ for $j = 1, \dots, N$, then $\Phi \circ (v, h)$ is a Blaschke product of degree $d(vp) - N$.*

Proof. By Theorem 7.12, the function $\Phi \circ (v, h)$ has one cancellation at each point ζ_j , $j = 1, \dots, N$.

For a given rational Γ -inner $h = (s, p)$, what can we expect of the \mathcal{E} -classes to which h belongs, in terms of the degree $d(p)$? For every $\nu \geq 0$, we trivially have $h \in \mathcal{E}_{\nu, \nu+d(p)+1}$; the interesting question is whether $h \in \mathcal{E}_{\nu k}$ for some k less than $\nu + d(p) + 1$. By Proposition 7.13, if h is not full the answer is no. If h is full we can always arrange one cancellation at a royal node, and so $h \in \mathcal{E}_{\nu, \nu+d(p)}$ for every $\nu \geq 0$. If h has N royal nodes in \mathbb{T} we can arrange N cancellations in $\Phi \circ (m, h)$ by choosing ν sufficiently large that there exists $m \in \mathcal{B}l_\nu$ that maps the royal nodes of h in \mathbb{T} to the required target points in \mathbb{T} , as in Corollary 7.15, and then we shall have $h \in \mathcal{E}_{\nu, \nu+d(p)-N+1}$. However, the question as to how large ν must be is subtle, as we shall see in the following two examples; they show that, even for the symmetrization of a pair of Blaschke products, it is a delicate issue whether one can achieve 3 cancellations in $\Phi \circ (v, h)$ with $v \in \mathcal{B}l_1$.

Example 7.16. *Let ψ be a rational inner function on \mathbb{D} . The rational Γ -inner function*

$$(7.10) \quad h_\psi(\lambda) = (\lambda + \lambda\psi(\lambda), \lambda^2\psi(\lambda)), \quad \lambda \in \mathbb{D},$$

lies in $\mathcal{E}_{1, d(\psi)+2} \setminus \mathcal{E}_{1, d(\psi)+1}$.

Proof. By Theorem 7.12, for any Blaschke product v , the function $\Phi \circ (v, h_\psi)$ has a cancellation at ζ if and only if $\zeta \in \mathbb{T}$, ζ is a royal node for h_ψ and $v(\zeta) = \frac{1}{2}s_\psi(\zeta)$. The royal nodes for $h_\psi = (s_\psi, p_\psi)$ are the roots of $s_\psi^2(\lambda) - 4p_\psi(\lambda) = 0$, that is, $(\lambda - \lambda\psi(\lambda))^2 = 0$. Therefore the royal nodes ω_j for h_ψ on \mathbb{T} are the roots of $\psi(\lambda) = 1$, that is, $\omega_j \in \mathbb{T}$ such that $\psi(\omega_j) = 1$, $j = 1, \dots, d(\psi)$. For $j = 1, \dots, d(\psi)$, we have $\frac{1}{2}s_\psi(\omega_j) = \frac{1}{2}(\omega_j + \omega_j\psi(\omega_j)) = \overline{\omega_j}$.

To prove that h_ψ is not in $\mathcal{E}_{1, d(\psi)+1}$ we must show that, for all $m \in \mathcal{B}l_1$, the rational function $\Phi \circ (m, h_\psi)$ is not in $\mathcal{B}l_{d(\psi)}$. It is enough to show that, for all $m \in \mathcal{B}l_1$, the rational function $\Phi \circ (m, h_\psi)$ cannot have 3 cancellations. To get 3 cancellations we need 3 royal nodes $\omega_{j_1}, \omega_{j_2}$ and ω_{j_3} , $1 \leq j_1 < j_2 < j_3 \leq d(\psi)$, with $m(\omega_{j_1}) = \overline{\omega_{j_1}}$, $m(\omega_{j_2}) = \overline{\omega_{j_2}}$ and $m(\omega_{j_3}) = \overline{\omega_{j_3}}$. This is impossible since the points $\overline{\omega_{j_1}}, \overline{\omega_{j_2}}, \overline{\omega_{j_3}}$ are in the opposite cyclic order to $\omega_{j_1}, \omega_{j_2}, \omega_{j_3}$ on \mathbb{T} .

Let us show that $h_\psi \in \mathcal{E}_{0, d(\psi)+2}$, that is, there is $\omega \in \mathbb{T}$ such that $\Phi \circ (\omega, h_\psi) \in \mathcal{B}l_{d(\psi)+1}$. Take $\omega = \frac{1}{2}s_\psi(\omega_1)$; then, by Theorem 7.12, $\Phi \circ (\omega, h_\psi)$ has exactly one cancellation at ω_1 . Therefore $\Phi \circ (\omega, h_\psi)$ has degree $d(\psi) + 1$, and so $\Phi \circ (\omega, h_\psi) \in \mathcal{B}l_{d(\psi)+1}$. Note that $\mathcal{E}_{0, d(\psi)+2} \subset \mathcal{E}_{1, d(\psi)+2}$. Therefore $h_\psi \in \mathcal{E}_{1, d(\psi)+2} \setminus \mathcal{E}_{1, d(\psi)+1}$.

The next example looks similar to h_ψ , and yet here one *can* achieve 3 cancellations with v of degree 1.

Example 7.17. *For any positive integer j the rational Γ -inner function*

$$(7.11) \quad h_j(\lambda) = (\lambda^2 + \lambda^{2j+3}, \lambda^{2j+5}), \quad \lambda \in \mathbb{D},$$

belongs to $\mathcal{E}_{1, 2j+4} \setminus \mathcal{E}_{0, 2j+4}$.

Proof. By Example 6.2, h_j is Γ -inner. By Theorem 7.12, for any Blaschke product v , the function

$$\Phi \circ (v, h_j) = \frac{2vp_j - s_j}{2 - vs_j}$$

has a cancellation at ζ if and only if $\zeta \in \mathbb{T}$, ζ is a royal node for h_j and $v(\zeta) = \frac{1}{2}\overline{s_j(\zeta)}$. The royal nodes for $h_j = (s_j, p_j)$ are the roots of $s_j^2(\lambda) - 4p_j(\lambda) = 0$, that is, $(\lambda^2 - \lambda^{2j+3})^2 = 0$. Therefore the royal nodes ω_k for h_j on \mathbb{T} are the $(2j+1)$ th roots of 1, that is,

$$\omega_k = e^{2i\pi k/(2j+1)}, \quad k = 0, \dots, 2j.$$

To prove that h_j is not in $\mathcal{E}_{0,2j+4}$ we must show that, for all $\omega \in \mathbb{T}$, the rational function $\Phi \circ (\omega, h_j)$ is not in $\mathcal{B}l_{2j+3}$. If $\omega = \frac{1}{2}\overline{s_j(\omega_k)}$ for some $k = 0, \dots, 2j$, then, by Theorem 7.12, $\Phi \circ (\omega, h_j)$ has exactly one cancellation at ω_k . Therefore $\Phi \circ (\omega, h_j)$ has degree $2j+4$, and so $\Phi \circ (\omega, h_j)$ is not in $\mathcal{B}l_{2j+3}$. If $\omega \neq \frac{1}{2}\overline{s_j(\omega_k)}$ for all $k = 0, \dots, 2j$, then $\Phi \circ (\omega, h_j)$ has degree $2j+5$. Thus h_j is not in $\mathcal{E}_{0,2j+4}$.

Let us show that $h_j \in \mathcal{E}_{1,2j+4}$, that is, there is $m \in \mathcal{B}l_1$ such that

$$\Phi \circ (m, h_j) = \frac{2mp_j - s_j}{2 - ms_j} \in \mathcal{B}l_{2j+3}.$$

For all $k = 0, \dots, 2j$, $\frac{1}{2}\overline{s_j(\omega_k)} = \frac{1}{2}(\overline{\omega_k^2 + \omega_k^{2j+3}}) = \overline{\omega_k^2}$. Note that the points $\omega_0 = 1$, ω_1, ω_1^{j+1} have the same cyclic order as $\overline{\omega_0^2} = 1$, $\overline{\omega_1^2}$, $\overline{\omega_1^{j+1}^2} = \overline{\omega_1}$ on \mathbb{T} . Take $m \in \mathcal{B}l_1$ such that

$$m(\zeta) = \frac{1}{2}\overline{s_j(\zeta)} = \overline{\zeta^2}$$

for $\zeta = \omega_0, \omega_1, \omega_1^{j+1}$. By Corollary 7.15, $\Phi \circ (m, h)$ has cancellations at $\omega_0, \omega_1, \omega_1^{j+1}$. Thus $\Phi \circ (m, h_j)$ is a Blaschke product of degree $d(p) + d(m) - 3 = 2j+3$. Therefore $h_j \in \mathcal{E}_{1,2j+4} \setminus \mathcal{E}_{0,2j+4}$.

8. SUPERFICIAL Γ -INNER FUNCTIONS AND THE CLASSES $\mathcal{E}_{\nu 1}$

In the next three sections of the paper we derive some further information about the two-dimensional array of classes $(\mathcal{E}_{\nu k})$.

For any inner function φ and $\omega \in \mathbb{T}$ the function $h = (\omega + \varphi, \omega\varphi)$ is Γ -inner, and has the property that $h(\lambda)$ lies in the topological boundary $\partial\Gamma$ of Γ for all $\lambda \in \mathbb{D}$. We shall prove not only a converse of this statement, but also the fact that all the classes in the first column of the array $(\mathcal{E}_{\nu k})$ consist of precisely this type of Γ -inner function.

Definition 8.1. A function $h \in \text{Hol}(\mathbb{D}, \Gamma)$ is superficial if $h(\mathbb{D}) \subset \partial\Gamma$.

Lemma 8.2. Let h be a nonconstant Γ -inner function such that

$$\Phi_\omega \circ h = \kappa$$

identically on \mathbb{D} for some $\omega, \kappa \in \mathbb{T}$. Then $\kappa = -\bar{\omega}$ and $h = (\omega p + \bar{\omega}, p)$ for some inner function p .

Proof. Let $h = (s, p)$: then p is a nonconstant inner function. By assumption,

$$\Phi_\omega(s, p) = \frac{2\omega p - s}{2 - \omega s} = \kappa.$$

Hence

$$s = 2\frac{\omega p - \kappa}{1 - \omega \kappa}.$$

Since $|s| \leq 2$, we have $|1 - \omega\kappa| \geq |\omega p - \kappa|$. Therefore for all $\lambda \in \Delta$,

$$|1 - \omega\kappa| \geq |\omega p(\lambda) - \kappa| = |p(\lambda) - \bar{\omega}\kappa|.$$

Since p is nonconstant there is $\lambda_0 \in \mathbb{T}$ such that $p(\lambda_0) = -\bar{\omega}\kappa$, and so $|1 - \omega\kappa| \geq 2$. Thus we have $\omega\kappa = -1$ and $\kappa = -\bar{\omega}$. Hence

$$s = 2 \frac{\omega p - \kappa}{1 - \omega\kappa} = \omega p + \bar{\omega}.$$

Proposition 8.3. *A Γ -inner function h is superficial if and only if there is an $\omega \in \mathbb{T}$ and an inner function p such that $h = (\omega p + \bar{\omega}, p)$.*

Proof. \Leftarrow Since the topological boundary $\partial\Gamma$ of Γ comprises the points $\{(z + w, zw) : |z| = 1, |w| \leq 1\}$, it follows that $h(\mathbb{D}) \subset \partial\Gamma$.

\Rightarrow Consider a superficial Γ -inner $h = (s, p)$; necessarily p is inner. If h is constant we may write $(s, p) = (\omega + z, \omega z)$ for some $\omega \in \mathbb{T}$ and $z \in \Delta$; then we have $z = \bar{\omega}p$ and $h = (\omega p + \bar{\omega}, p)$.

Now suppose that h is nonconstant. It follows that s is nonconstant, and hence $|s(0)| < 2$. Observe that, for any $(s_0, p_0) \in \partial\Gamma$ we have $|s_0 - \bar{s}_0 p_0| = 1 - |p_0|^2$. Now $h(0) \in \partial\Gamma$, and so there exists $\omega \in \mathbb{T}$ such that

$$\omega \left(s(0) - \overline{s(0)} p(0) \right) = |s(0) - \overline{s(0)} p(0)| = 1 - |p(0)|^2.$$

A simple calculation (or [6, Theorem 2.5]) shows that

$$|\Phi_\omega(s(0), p(0))| = \left| \frac{2\omega p(0) - s(0)}{2 - \omega s(0)} \right| = 1.$$

Therefore $\Phi_\omega \circ h$ is an inner function which takes a value of modulus 1 at 0, and so it is a constant function, with value κ , say. By Lemma 8.2, $\kappa = -\bar{\omega}$ and $s = \omega p + \bar{\omega}$, and so $h = (\omega p + \bar{\omega}, p)$.

The image of a function in $\text{Hol}(\mathbb{D}, \Gamma)$ is either contained in or disjoint from $\partial\Gamma$.

Lemma 8.4. *If $h \in \text{Hol}(\mathbb{D}, \Gamma)$ is not superficial then $h(\mathbb{D}) \subset \mathbb{G}$.*

Proof. Let $h = (s, p)$. Suppose that $h(\mathbb{D})$ is not contained in \mathbb{G} : then there exists $\lambda_0 \in \mathbb{D}$ such that $h(\lambda_0) \in \partial\Gamma$. Let $u(\lambda)$ be the spectral radius of the matrix

$$\begin{pmatrix} 0 & 1 \\ -p(\lambda) & s(\lambda) \end{pmatrix}$$

for $\lambda \in \mathbb{D}$. By Vesentini's Theorem [14, Theorem 2.3.32], u is subharmonic in \mathbb{D} . Since $u(\lambda)$ is the maximum of the moduli of the roots of the equation $z^2 - s(\lambda)z + p(\lambda) = 0$, we have

$$u(\lambda) = \max\{|z|, |w|\} \text{ where } s(\lambda) = z + w, p(\lambda) = zw.$$

Since $(s(\lambda), p(\lambda)) \in \Gamma$, we have $0 \leq u(\lambda) \leq 1$ for all $\lambda \in \mathbb{D}$, and $u(\lambda) = 1$ if and only if $(s(\lambda), p(\lambda)) \in \partial\Gamma$. By hypothesis u attains its maximum at a point $\lambda_0 \in \mathbb{D}$, and hence u is constant and equal to 1 on \mathbb{D} . Consequently $h(\lambda) \in \partial\Gamma$ for all $\lambda \in \mathbb{D}$.

Proposition 8.5. *The class \mathcal{E}_{01} consists of the superficial rational Γ -inner functions.*

Proof. By definition, a rational function $f = (s, p) \in \mathcal{E}_{01}$ if and only if there are $\omega, \kappa \in \mathbb{T}$ such that

$$(8.1) \quad \Phi_\omega \circ (s, p) = \frac{2\omega p - s}{2 - \omega s} = \kappa.$$

First consider the case that p is nonconstant. For $f = (s, p)$, the conditions of Lemma 8.2 are satisfied. Hence, we have $\kappa = -\bar{\omega}$, $s = \omega p + \bar{\omega}$ and $f = (\omega p + \bar{\omega}, p)$. By Proposition 8.3, f is a superficial Γ -inner function.

In the case that p is constant, since $f = (s, p)$ is a Γ -inner function, by Proposition 3.2, $|s| \leq 2$, $|p| = 1$, $s = \bar{s}p$ and $f = (s, p) \in b\Gamma$, and so is superficial.

Theorem 8.6. *For every $\nu \geq 1$, the class $\mathcal{E}_{\nu 1}$ is equal to \mathcal{E}_{01} and consists of the superficial rational Γ -inner functions.*

Proof. By Definition 7.1 and Theorem 7.3, the function $h = (s, p) \in \mathcal{E}_{\nu 1}$ if $h = (s, p)$ is rational Γ -inner and there exists $m \in \mathcal{B}l_\nu$ such that

$$\frac{2mp - s}{2 - ms} = \kappa \in \mathbb{T}.$$

Then

$$s = 2 \frac{mp - \kappa}{1 - m\kappa}.$$

Since $|s| \leq 2$ on \mathbb{T} , we have

$$|mp - \kappa|^2 \leq |1 - m\kappa|^2.$$

Since m, p and κ are all unimodular on \mathbb{T} , on expanding we find that

$$1 - 2\operatorname{Re}(mp\bar{\kappa}) + 1 \leq 1 - 2\operatorname{Re}(m\kappa) + 1,$$

on \mathbb{T} , and so

$$(8.2) \quad \operatorname{Re}(m\kappa)(\lambda) \leq \operatorname{Re}(mp\bar{\kappa})(\lambda)$$

for all $\lambda \in \mathbb{T}$.

Therefore, if $m\kappa = 1$ at some point $\lambda \in \mathbb{T}$ then $mp\bar{\kappa} = 1$ at the point λ . Suppose the Blaschke product m has degree $\nu' \leq \nu$, so that $m\kappa = 1$ at ν' distinct points $\lambda_1, \dots, \lambda_{\nu'} \in \mathbb{T}$. Then also $mp\bar{\kappa} = 1$ at the points $\lambda_1, \dots, \lambda_{\nu'} \in \mathbb{T}$. Hence

$$(mp\bar{\kappa})(\lambda_j) = (m\kappa)(\lambda_j)(\bar{\kappa}^2 p)(\lambda_j) = \bar{\kappa}^2 p(\lambda_j) = 1$$

for $j = 1, \dots, \nu'$. Thus $p = \kappa^2$ at ν' distinct points $\lambda_1, \dots, \lambda_{\nu'} \in \mathbb{T}$, and so the degree $d(p) \geq d(m) = \nu'$.

We claim that $d(m) = 0$. Suppose $d(m) > 0$; then $d(p) \geq d(m) > 0$, and p is nonconstant. Since p is nonconstant and rational, by Proposition 7.7, the phasar derivative $A(p)$ of p is strictly positive on \mathbb{T} . Observe that phasar derivatives satisfies the following inequality

$$A(mp\bar{\kappa}) = A(mp) = A(m) + A(p) > A(m) = A(m\kappa)$$

on \mathbb{T} . Recall that $|mp\bar{\kappa}| = 1 = |m\kappa| = 1$ on \mathbb{T} , and, as we have shown above, there is a point $\lambda_0 \in \mathbb{T}$ such that $m\kappa(\lambda_0) = 1$ and $mp\bar{\kappa}(\lambda_0) = 1$. Therefore, at some $\lambda' \in \mathbb{T}$ close to the point λ_0 , we have

$$\operatorname{Re}(mp\bar{\kappa})(\lambda') < \operatorname{Re}(m\kappa)(\lambda')$$

which is a contradiction to the inequality (8.2). Hence $d(m) = 0$, and therefore, $h = (s, p) \in \mathcal{E}_{01}$.

9. THE CLASSES $\mathcal{E}_{\nu k}$ AND k -EXTREMALS

In this section we show that the elements of $\mathcal{E}_{\nu k}$ are k -extremal for Γ and that, if the Γ -interpolation Conjecture 4.1 is true, then every k -extremal rational Γ -inner function belongs to $\mathcal{E}_{k-2, k}$.

For $\zeta \in \mathbb{C}$ and $(s, p) \in \mathbb{C}^2$ we define

$$\zeta \cdot (s, p) = (\zeta s, \zeta^2 p) \quad \text{and} \quad \zeta \cdot \Gamma = \{\zeta \cdot z : z \in \Gamma\}.$$

Theorem 9.1. *If $h \in \mathcal{E}_{\nu k}$, where $\nu \geq 0$ and $k \geq 2$, and h is not superficial then h is k -extremal for $\text{Hol}(\mathbb{D}, \Gamma)$.*

Proof. Let $h \in \mathcal{E}_{\nu k}$ be not superficial. There exist $m \in \mathcal{B}l_\nu$ and $q \in \mathcal{B}l_{k-1}$ such that $\Phi \circ (m, h) = q$. Suppose that h is not k -extremal: then there exist k distinct points $\lambda_1, \dots, \lambda_k \in \mathbb{D}$, an $r_0 > 1$ and a function $f \in \text{Hol}(r_0\mathbb{D}, \Gamma)$ such that $f(\lambda_j) = h(\lambda_j)$ for $j = 1, \dots, k$. Since h is not superficial, the points $h(\lambda_j) \in \mathbb{G}$, by Lemma 8.4, and so, by the same lemma, $f(r_0\mathbb{D}) \subset \mathbb{G}$.

Pick any r_1 in the interval $(1, r_0)$: then $f(r_1\Delta)$ is a compact subset of \mathbb{G} . Now

$$f(r_1\Delta) \subset \bigcup_{0 < \rho < 1} \rho \cdot \mathbb{G} = \mathbb{G},$$

and hence there exists $\rho \in (0, 1)$ such that $f(r_1\Delta) \subset \rho \cdot \mathbb{G} \subset \rho \cdot \Gamma$.

Observe that, for $\lambda \in \Delta$ and $(s, p) \in \Gamma$ we have

$$\Phi(\lambda, \rho \cdot (s, p)) = \Phi(\lambda, \rho s, \rho^2 p) = \frac{2\lambda\rho^2 p - \rho s}{2 - \lambda\rho s} = \rho\Phi(\rho\lambda, s, p) \in \rho\Delta.$$

Thus

$$\Phi(\Delta \times \rho \cdot \Gamma) \subset \rho\Delta \subset \mathbb{D}.$$

Furthermore, Φ is analytic on $(\rho^{-1}\mathbb{D}) \times \rho \cdot \Gamma$. Hence, by continuity of Φ and compactness of $\rho \cdot \Gamma$, there is a neighbourhood U of Δ such that

$$\Phi(U \times \rho \cdot \Gamma) \subset \mathbb{D}.$$

Pick r_2 in the interval $(1, r_1)$ such that $m(r_2\mathbb{D}) \subset U$. Then for all $\lambda \in r_2\mathbb{D}$ and $z \in \rho \cdot \Gamma$,

$$|\Phi(m(\lambda), z)| < 1.$$

In particular, for all $\lambda \in r_2\mathbb{D}$,

$$(9.1) \quad |\Phi(m(\lambda), f(\lambda))| < 1.$$

Thus $\Phi \circ (m, f)$ belongs to the Schur class, and

$$\Phi \circ (m, f)(\lambda_j) = \Phi \circ (m, h)(\lambda_j) = q(\lambda_j) \quad \text{for } j = 1, \dots, k.$$

Hence $\Phi \circ (m, f)$ is a solution of the solvable Nevanlinna-Pick problem

$$\lambda_j \mapsto q(\lambda_j), \quad j = 1, \dots, k,$$

as is $q \in \mathcal{B}l_{k-1}$. Any k -point Nevanlinna-Pick problem that is solved by an element of $\mathcal{B}l_{k-1}$ is extremally solvable and has a unique solution, and so $\Phi \circ (m, f) = q$.

This is a contradiction, since, by inequality (9.1), $\Phi \circ (m, f)$ maps $r_2\mathbb{D}$ into \mathbb{D} , whereas q maps $r_2\mathbb{D} \setminus \Delta$ to the complement of Δ . Hence h is k -extremal.

If Conjecture 4.1 is true then all n -extremals for Γ lie in $\mathcal{E}_{n-2,n}$.

Observation 9.2. *Let $n \geq 2$. If condition \mathcal{C}_{n-2} suffices for the solvability of n -point Γ -interpolation problems then every rational Γ -inner function h which is n -extremal for $\text{Hol}(\mathbb{D}, \Gamma)$ belongs to $\mathcal{E}_{n-2,n}$.*

Proof. Let h be n -extremal for $\text{Hol}(\mathbb{D}, \Gamma)$ and suppose that $h \notin \mathcal{E}_{n-2,n}$. Thus for every $v \in \mathcal{B}l_{n-2}$ the function $\Phi \circ (v, h) \notin \mathcal{B}l_{n-1}$.

Consider n distinct points $\lambda_1, \dots, \lambda_n$ in \mathbb{D} and let $z_j = h(\lambda_j)$, $1 \leq j \leq n$. Then, for all $v \in \mathcal{B}l_{n-2}$, the Nevanlinna-Pick data

$$(9.2) \quad \lambda_j \mapsto \Phi(v(\lambda_j), z_j), \quad 1 \leq j \leq n,$$

are not extremally solvable, so that $\|X(v)\| < 1$, where the operator $X(v)$ is defined on

$$(9.3) \quad \mathcal{M} = \text{span} \{K_{\lambda_1}, \dots, K_{\lambda_n}\} \subset H^2,$$

by

$$(9.4) \quad X(v)K_{\lambda_j} = \overline{\Phi(v(\lambda_j), z_j)}K_{\lambda_j}, \quad 1 \leq j \leq n.$$

Thus, by the compactness of $\mathcal{B}l_{n-2}$ and the continuity of $X(\cdot)$, there exists a positive constant $c < 1$ such that

$$(9.5) \quad \sup_{v \in \mathcal{B}l_{n-2}} \|X(v)\| \leq c < 1.$$

For $r \geq 1$ define the operator $X_r(v)$ on

$$(9.6) \quad \mathcal{M}_r \stackrel{\text{def}}{=} \text{span} \{K_{\lambda_1/r}, \dots, K_{\lambda_n/r}\} \subset H^2,$$

by

$$(9.7) \quad X_r(v)K_{\lambda_j/r} = \overline{\Phi(v(\lambda_j/r), z_j)}K_{\lambda_j/r}, \quad 1 \leq j \leq n.$$

Lemma 9.3. *There exists $r > 1$ such that*

$$\sup_{v \in \mathcal{B}l_{n-2}} \|X_r(v)\| < 1.$$

Proof. Note that

$$(9.8) \quad X_r(v) = T_r Y_r(v) T_r^{-1}$$

where

$$(9.9) \quad T_r : \mathcal{M} \rightarrow \mathcal{M}_r : K_{\lambda_j} \mapsto K_{\lambda_j/r}, \quad 1 \leq j \leq n.$$

and

$$(9.10) \quad Y_r(v) : \mathcal{M} \rightarrow \mathcal{M} : K_{\lambda_j} \mapsto \overline{\Phi(v(\lambda_j/r), z_j)}K_{\lambda_j}, \quad 1 \leq j \leq n.$$

Since $Y_r(v)$ is a diagonal operator on \mathcal{M} (with respect to a fixed basis) and its diagonal entries are continuous in $r \geq 1$, $v \in \mathcal{B}l_{n-2}$, the map $(r, v) \mapsto Y_r(v)$ is a continuous $\mathcal{L}(\mathcal{M})$ -valued map on $[1, \infty) \times \mathcal{B}l_{n-2}$. Note that $Y_1(v) = X(v)$ and $\|X(v)\| \leq c < 1$ for all $v \in \mathcal{B}l_{n-2}$. Thus there is $r_0 > 1$ such that

$$\|Y_r(v)\| \leq \frac{1}{2}(c + 1) < 1$$

for all $v \in \mathcal{B}l_{n-2}$ and $1 < r < r_0$. Therefore

$$\|X_r(v)\| = \|T_r Y_r(v) T_r^{-1}\| \leq \frac{1}{2}(c+1) \|T_r\| \|T_r^{-1}\|$$

for all $v \in \mathcal{B}l_{n-2}$ and $1 < r < r_0$. Since $K_{\lambda_j/r} \rightarrow K_{\lambda_j}$ in H^2 as $r \rightarrow 1$, it is straightforward to show that $\|T_r\| \rightarrow 1$ and $\|T_r^{-1}\| \rightarrow 1$ as $r \rightarrow 1$. Hence, for $r > 1$ sufficiently close to 1, we have $\|X_r(v)\| < 1$ for all $v \in \mathcal{B}l_{n-2}$.

Conclusion of the Proof of Observation 9.2. By Lemma 9.3, there exists $r > 1$ such that $\|X_r(v)\| < 1$ for all $v \in \mathcal{B}l_{n-2}$. Therefore, for every Blaschke product $v \in \mathcal{B}l_{n-2}$, the Nevanlinna-Pick data

$$(9.11) \quad \lambda_j/r \mapsto \Phi(v(\lambda_j/r), z_j), \quad 1 \leq j \leq n,$$

are solvable. In other words, $\lambda_j/r \mapsto z_j$, $1 \leq j \leq n$, satisfy the condition \mathcal{C}_{n-2} .

By assumption, \mathcal{C}_{n-2} suffices for solvability of n -point Γ -interpolation problems. Therefore there exists $f \in \text{Hol}(\mathbb{D}, \Gamma)$ such that

$$f(\lambda_j/r) = z_j, \quad 1 \leq j \leq n.$$

Then the function $g(\lambda) = f(\lambda/r)$ belongs to $\text{Hol}(r\mathbb{D}, \Gamma)$ and satisfies

$$g(\lambda_j) = f(\lambda_j/r) = z_j = h(\lambda_j), \quad 1 \leq j \leq n.$$

This contradicts the n -extremality of h . Thus $h \in \mathcal{E}_{n-2,n}$.

10. COMPLEX GEODESICS OF \mathbb{G} AND THE CLASSES $\mathcal{E}_{\nu 2}$

In this section we shall show that all the classes $\mathcal{E}_{\nu 2}$ consist of the superficial rational Γ -inner functions together with the complex geodesics of \mathbb{G} . First we recall a result from [7].

Proposition 10.1. *An analytic function $h : \mathbb{D} \rightarrow \mathbb{G}$ is a complex geodesic of \mathbb{G} if and only if there is an $\omega \in \mathbb{T}$ such that $\Phi_\omega \circ h \in \text{Aut } \mathbb{D}$. Furthermore, every complex geodesic of \mathbb{G} is Γ -inner.*

Proof. \Leftarrow Suppose there is $\omega \in \mathbb{T}$ such that $\Phi_\omega \circ h \in \text{Aut } \mathbb{D}$, say $\Phi_\omega \circ h = v \in \text{Aut } \mathbb{D}$. Then $g = v^{-1} \circ \Phi_\omega : \mathbb{G} \rightarrow \mathbb{D}$ is an analytic left inverse of h . Therefore, h is a complex geodesic of \mathbb{G} .

\Rightarrow Let h be a complex geodesic of \mathbb{G} and let $g : \mathbb{G} \rightarrow \mathbb{D}$ be an analytic left inverse of h . For any distinct points $\lambda_1, \lambda_2 \in \mathbb{D}$, we have

$$C_{\mathbb{G}}(h(\lambda_1), h(\lambda_2)) \leq \rho(\lambda_1, \lambda_2).$$

On the other hand, since $g \circ h = \text{id}_{\mathbb{D}}$,

$$\rho(\lambda_1, \lambda_2) = \rho(g \circ h(\lambda_1), g \circ h(\lambda_2)) \leq C_{\mathbb{G}}(h(\lambda_1), h(\lambda_2)).$$

Therefore,

$$C_{\mathbb{G}}(h(\lambda_1), h(\lambda_2)) = \rho(\lambda_1, \lambda_2).$$

By [7, Theorem 1.2], there is $\omega \in \mathbb{T}$ such that $\Phi_\omega \circ h \in \text{Aut } \mathbb{D}$.

By [7, Lemma 1.1], h is Γ -inner.

Corollary 10.2. *The set $\mathcal{E}_{02} \setminus \mathcal{E}_{01}$ is precisely the set of complex geodesics of \mathbb{G} .*

Proof. Let $h \in \mathcal{E}_{02} \setminus \mathcal{E}_{01}$. By Definition 7.1, there exists $\omega \in \mathbb{T}$ such that

$$\Phi_\omega \circ h = \frac{2\omega p - s}{2 - \omega s} \in \mathcal{B}l_1 \setminus \mathcal{B}l_0 = \text{Aut } \mathbb{D}.$$

Hence, by Proposition 10.1, h is a complex geodesic of \mathbb{G} .

Conversely, suppose that h is a complex geodesic of \mathbb{G} . By Proposition 10.1, there exists $\omega \in \mathbb{T}$ such that $\Phi_\omega \circ h \in \text{Aut } \mathbb{D}$, and hence $h \in \mathcal{E}_{02}$. Since $h(\mathbb{D}) \subset \mathbb{G}$, h is not superficial, and so $h \notin \mathcal{E}_{01}$.

Theorem 10.3. *For $\nu \geq 0$ the set $\mathcal{E}_{\nu 2}$ is the union of the set of superficial rational Γ -inner functions and the set of complex geodesics of \mathbb{G} .*

Proof. Since $\mathcal{E}_{02} \subset \mathcal{E}_{\nu 2}$, it follows from Corollary 10.2 that $\mathcal{E}_{\nu 2}$ contains all superficial rational Γ -inner functions and all complex geodesics of \mathbb{G} . If $h \in \mathcal{E}_{\nu 2}$ then, by Lemma 8.4, either h is superficial or $h \in \text{Hol}(\mathbb{D}, \mathbb{G})$. In the latter case, h is 2-extremal by Theorem 9.1 and h is a complex geodesic of \mathbb{G} by Corollary 2.7.

11. CONDITION \mathcal{C}_ν AND THE CLASSES $\mathcal{E}_{\nu k}$

It is clear that $\mathcal{C}_\nu(\lambda, z)$ implies $\mathcal{C}_{\nu-1}(\lambda, z)$ for any Γ -interpolation data $\lambda \mapsto z$. To show that \mathcal{C}_ν is *strictly stronger* than $\mathcal{C}_{\nu-1}$ we need to find data

$$(11.1) \quad \lambda = (\lambda_1, \dots, \lambda_k), \quad z = (z_1, \dots, z_k),$$

where $\lambda_1, \dots, \lambda_k$ are distinct points in \mathbb{D} and $z_j = (s_j, p_j) \in \mathbb{G}$ for $j = 1, \dots, k$, such that

(i) for every Blaschke product v of degree at most $\nu - 1$,

$$(11.2) \quad \lambda_j \mapsto \frac{2v(\lambda_j)p_j - s_j}{2 - v(\lambda_j)s_j}, \quad j = 1, \dots, k,$$

are solvable Nevanlinna-Pick data, but

(ii) there is a Blaschke product m of degree ν such that

$$(11.3) \quad \lambda_j \mapsto \frac{2m(\lambda_j)p_j - s_j}{2 - m(\lambda_j)s_j}, \quad j = 1, \dots, k,$$

are not solvable Nevanlinna-Pick data.

For distinct points $\lambda_1, \dots, \lambda_k$ in \mathbb{D} , we define

$$\text{Solv}(\lambda_1, \dots, \lambda_k) = \{(f(\lambda_1), \dots, f(\lambda_k)) \in \mathbb{D}^k : f \in \mathcal{S}\},$$

and

$$\text{Unsolv}(\lambda_1, \dots, \lambda_k) = \mathbb{C}^k \setminus \text{Solv}(\lambda_1, \dots, \lambda_k).$$

Thus $w = (w_1, \dots, w_k) \in \text{Solv}(\lambda_1, \dots, \lambda_k)$ if and only if $\lambda_j \mapsto w_j$, $j = 1, \dots, k$, are solvable Nevanlinna-Pick data.

Proposition 11.1. *Let $\lambda_1, \dots, \lambda_n$ be distinct points in \mathbb{D} .*

(i) *$\text{Solv}(\lambda_1, \dots, \lambda_n)$ is closed in \mathbb{C}^n .*

(ii) *Let $w = (w_1, \dots, w_n) \in \text{Solv}(\lambda_1, \dots, \lambda_n)$. The Nevanlinna-Pick data $\lambda_j \mapsto w_j$, $j = 1, \dots, n$, are extremally solvable if and only if $w \in \partial \text{Solv}(\lambda_1, \dots, \lambda_n)$.*

Proof. Part (i) is immediate from Pick's Theorem, which asserts that $w = (w_1, \dots, w_n) \in \text{Solv}(\lambda_1, \dots, \lambda_n)$ if and only if

$$\left[\frac{1 - \bar{w}_i w_j}{1 - \bar{\lambda}_i \lambda_j} \right]_{i,j=1}^n \geq 0.$$

(ii) Suppose that the Nevanlinna-Pick data $\lambda_j \mapsto w_j$, $j = 1, \dots, n$, are extremally solvable. We will show that $w \in \partial\text{Solv}(\lambda_1, \dots, \lambda_n)$ by induction on n . It is true when $n = 1$, since $\lambda_1 \mapsto w_1$ is an extremally solvable Nevanlinna-Pick datum if and only if $|w_1| = 1$. Consider $n \geq 2$ and suppose (ii) holds for $n - 1$. Let $\lambda_j \mapsto w_j$, $1 \leq j \leq n$, be extremally solvable Nevanlinna-Pick data, and let $f \in \mathcal{S}$ be a solution. The Schur reduction f_1 of f at λ_1 ,

$$(11.4) \quad f_1(\lambda) = \frac{1 - \bar{\lambda}_1 \lambda}{\lambda - \lambda_1} \frac{f(\lambda) - w_1}{1 - \bar{w}_1 f(\lambda)},$$

also lies in \mathcal{S} and satisfies $f_1(\lambda_j) = w'_j$, $2 \leq j \leq n$, where

$$w'_j = \frac{1 - \bar{\lambda}_1 \lambda_j}{\lambda_j - \lambda_1} \frac{w_j - w_1}{1 - \bar{w}_1 w_j}.$$

We claim that the Nevanlinna-Pick data

$$(11.5) \quad \lambda_j \mapsto w'_j, \quad 2 \leq j \leq n,$$

are also extremally solvable. They are certainly solvable, since f_1 is a solution. If they are not extremally solvable then there are two distinct functions $f_1, \tilde{f}_1 \in \mathcal{S}$ that solve the data, and on inverting the relation (11.4) we obtain two distinct solutions of the Nevanlinna-Pick problem with data $\lambda_j \mapsto w_j$, $1 \leq j \leq n$; this contradicts the fact that extremally solvable Nevanlinna-Pick problems have unique solutions. The claim follows.

By the inductive hypothesis there is a sequence $(w'(k))_{k \geq 1}$ in $\text{Unsolv}(\lambda_2, \dots, \lambda_n)$ converging to $w' = (w'_2, \dots, w'_n)$ as $k \rightarrow \infty$. Let

$$w_j(k) = \frac{\frac{\lambda_j - \lambda_1}{1 - \bar{\lambda}_1 \lambda_j} w'_j(k) + w_1}{1 + \bar{w}_1 \frac{\lambda_j - \lambda_1}{1 - \bar{\lambda}_1 \lambda_j} w'_j(k)} \quad \text{for } k \geq 1, \quad 2 \leq j \leq n.$$

Then

$$w(k) \stackrel{\text{def}}{=} (w_1, w_2(k), \dots, w_n(k)), \quad k = 1, 2, \dots,$$

is a sequence in $\text{Unsolv}(\lambda_1, \lambda_2, \dots, \lambda_n)$ that converges to w as $k \rightarrow \infty$. Thus $w \in \partial\text{Solv}(\lambda_1, \lambda_2, \dots, \lambda_n)$.

Hence, by induction, $w \in \partial\text{Solv}(\lambda_1, \dots, \lambda_n)$ for all $n \in \mathbb{N}$.

Suppose the Nevanlinna-Pick data $\lambda_j \mapsto w_j$, $j = 1, \dots, n$, are not extremally solvable. Then

$$\left[\frac{1 - \bar{w}_i w_j}{1 - \bar{\lambda}_i \lambda_j} \right]_{i,j=1}^n > 0.$$

Note that $|w_j| < 1$ for all $j = 1, \dots, n$. Hence, by continuity, there are neighbourhoods U_1, \dots, U_n of w_1, \dots, w_n in \mathbb{D} such that, for all $z_1 \in U_1, \dots, z_n \in U_n$, we have

$$\left[\frac{1 - \bar{z}_i z_j}{1 - \bar{\lambda}_i \lambda_j} \right]_{i,j=1}^n > 0.$$

Hence $(z_1, z_2, \dots, z_n) \in \text{Solv}(\lambda_1, \dots, \lambda_n)$ for all $(z_1, z_2, \dots, z_n) \in U_1 \times \dots \times U_n$. That is, there is a neighbourhood $U_1 \times \dots \times U_n$ of (w_1, \dots, w_n) that is contained in $\text{Solv}(\lambda_1, \dots, \lambda_n)$. In other words (w_1, \dots, w_n) is an interior point of $\text{Solv}(\lambda_1, \dots, \lambda_n)$, hence is not in $\partial \text{Solv}(\lambda_1, \dots, \lambda_n)$.

Proposition 11.2. *If there exists a nonconstant function $h \in \mathcal{E}_{\nu k} \setminus \mathcal{E}_{\nu-1, k}$ then \mathcal{C}_ν is strictly stronger than $\mathcal{C}_{\nu-1}$. In fact there is a set of Γ -interpolation data $\lambda_j \mapsto z_j$ with k interpolation points which satisfies $\mathcal{C}_{\nu-1}$ but not \mathcal{C}_ν .*

It follows of course from Theorem 4.3 that the data $\lambda_j \mapsto z_j$ are not solvable.

Proof. Pick any k distinct points $\lambda_1, \dots, \lambda_k$ in \mathbb{D} and let $h(\lambda_j) = (s_j, p_j)$, $j = 1, \dots, k$. Since $h = (s, p) \notin \mathcal{E}_{\nu-1, k}$, there is no $v \in \mathcal{B}l_{\nu-1}$ such that

$$(11.6) \quad \frac{2vp - s}{2 - vs} \in \mathcal{B}l_{k-1}.$$

Consider any $v \in \mathcal{B}l_{\nu-1}$. The Nevanlinna-Pick data

$$(11.7) \quad \lambda_j \mapsto \Phi(v(\lambda_j), s_j, p_j) = \frac{2v(\lambda_j)p_j - s_j}{2 - v(\lambda_j)s_j}, \quad j = 1, \dots, k,$$

are solvable, since $\lambda \mapsto \Phi(v(\lambda), h(\lambda))$ is in \mathcal{S} and satisfies the interpolation condition (11.7). However, if the data (11.7) are extremally solvable, then the function $\Phi \circ (v, h)$ is a Blaschke product of degree at most $k-1$ [1, Theorem 6.15], contrary to the hypothesis that there is no $v \in \mathcal{B}l_{\nu-1}$ such that equation $\Phi \circ (v, h) \in \mathcal{B}l_{k-1}$. Therefore, for all $v \in \mathcal{B}l_{\nu-1}$, the data (11.7) are not extremally solvable and so (w_1, \dots, w_k) lies in the interior of $\text{Solv}(\lambda_1, \dots, \lambda_k)$, where $w_j = \Phi(v(\lambda_j), s_j, p_j)$. Recall from Section 5 that, by Pick's Theorem, the Nevanlinna-Pick data

$$(11.8) \quad \lambda_j \mapsto \Phi(v(\lambda_j), s_j, p_j), \quad 1 \leq j \leq k,$$

are solvable if and only if the operator $X(v)$ on \mathcal{M} given by

$$(11.9) \quad X(v)K_{\lambda_j} = \overline{\Phi(v(\lambda_j), s_j, p_j)}K_{\lambda_j}, \quad 1 \leq j \leq k,$$

is a contraction. The Nevanlinna-Pick data (11.8) are extremally solvable if and only if

$$\sup_v \|X(v)\| = 1$$

where the supremum is over all Blaschke products of degree at most $\nu-1$. Thus, for all $v \in \mathcal{B}l_{\nu-1}$,

$$\|X(v)\| < 1.$$

By the compactness of $\mathcal{B}l_{\nu-1}$ and the continuity of X , there is a positive constant $c < 1$ such that, for all $v \in \mathcal{B}l_{\nu-1}$,

$$\|X(v)\| \leq c < 1.$$

Hence there is a neighbourhood U of (s_1, \dots, s_k) in $(2\mathbb{D})^k$ such that, for all $(\tilde{s}_1, \dots, \tilde{s}_k) \in U$ and all $v \in \mathcal{B}l_{\nu-1}$,

$$(11.10) \quad \lambda_j \mapsto \Phi(v(\lambda_j), \tilde{s}_j, p_j) = \frac{2v(\lambda_j)p_j - \tilde{s}_j}{2 - v(\lambda_j)\tilde{s}_j}, \quad j = 1, \dots, k,$$

are solvable Nevanlinna-Pick data.

By assumption, $h = (s, p) \in \mathcal{E}_{\nu k}$, and so there is $m \in \mathcal{B}l_\nu$ such that

$$q \stackrel{\text{def}}{=} \Phi \circ (m, h) = \frac{2mp - s}{2 - ms} \in \mathcal{B}l_{k-1}.$$

Let

$$\tilde{w}_j \stackrel{\text{def}}{=} q(\lambda_j) = \Phi(m(\lambda_j), s_j, p_j) = \frac{2m(\lambda_j)p_j - s_j}{2 - m(\lambda_j)s_j}, \quad j = 1, \dots, k.$$

Then the Nevanlinna-Pick data

$$\lambda_j \mapsto \tilde{w}_j, \quad j = 1, \dots, k,$$

are solvable and q is a solution. By [1, Lemma 6.19], the Pick matrix

$$\left[\frac{1 - \overline{\tilde{w}_i} \tilde{w}_j}{1 - \overline{\lambda_i} \lambda_j} \right]_{i,j=1}^k \geq 0.$$

is positive and of rank at most $k - 1$. Hence the Pick matrix is singular and the data $\lambda_j \mapsto \tilde{w}_j$, $j = 1, \dots, k$, are extremally solvable. By Proposition 11.1,

$$\tilde{w} = (\Phi(m(\lambda_j), s_j, p_j))_{j=1}^k \in \partial \text{Solv}(\lambda_1, \dots, \lambda_k).$$

Define an analytic function

$$F : U \rightarrow \mathbb{D}^k : (\zeta_1, \dots, \zeta_k) \mapsto (\Phi(m(\lambda_j), \zeta_j, p_j))_{j=1}^k.$$

Note that $F(s_1, \dots, s_k) = \tilde{w} \in \partial \text{Solv}(\lambda_1, \dots, \lambda_k)$. The Jacobian matrix

$$J_F(s_1, \dots, s_k) = \text{diag} \left(2 \frac{m(\lambda_j)^2 p_j - 1}{(2 - m(\lambda_j)s_j)^2} \right)_{j=1}^k$$

is nonsingular. Hence, by the Inverse Function Theorem [15], there is a neighbourhood W of (s_1, \dots, s_k) in U such that $F(W)$ is an open neighbourhood of \tilde{w} in \mathbb{C}^k and $F|_W$ is bijective. Pick a point $w' \in F(W) \setminus \text{Solv}(\lambda_1, \dots, \lambda_k)$ and let $(\tilde{s}_1, \dots, \tilde{s}_k) = F^{-1}(w')$. Then $(\tilde{s}_1, \dots, \tilde{s}_k) \in W \subset U$, and so

$$(11.11) \quad \lambda_j \mapsto \frac{2v(\lambda_j)p_j - \tilde{s}_j}{2 - v(\lambda_j)\tilde{s}_j}, \quad j = 1, \dots, k,$$

are solvable Nevanlinna-Pick data for every Blaschke product $v \in \mathcal{B}l_{\nu-1}$. Thus $(\lambda_j, \tilde{s}_j, p_j)$, $j = 1, \dots, k$, satisfy $\mathcal{C}_{\nu-1}$.

On the other hand, since $F(\tilde{s}_1, \dots, \tilde{s}_k) = w' \in \text{Unsolv}(\lambda_1, \dots, \lambda_k)$,

$$(11.12) \quad \lambda_j \mapsto \frac{2m(\lambda_j)p_j - \tilde{s}_j}{2 - m(\lambda_j)\tilde{s}_j}, \quad j = 1, \dots, k,$$

are not solvable Nevanlinna-Pick data for $m \in \mathcal{B}l_\nu$, and thus $(\lambda_j, \tilde{s}_j, p_j)$, $j = 1, \dots, k$, do not satisfy \mathcal{C}_ν .

12. INEQUALITIES FOR THE CLASSES $\mathcal{E}_{\nu k}$

In order to apply Proposition 11.2 we must establish the strict inclusion

$$\mathcal{E}_{\nu-1,k} \subsetneq \mathcal{E}_{\nu,k}$$

for a suitable k .

Proposition 12.1. *For all $\nu \geq 1$ and $0 < r < 1$, the function*

$$(12.1) \quad h_\nu(\lambda) = \left(2(1-r) \frac{\lambda^{\nu+1}}{1+r\lambda^{2\nu+1}}, \frac{\lambda(\lambda^{2\nu+1}+r)}{1+r\lambda^{2\nu+1}} \right), \quad \lambda \in \mathbb{D},$$

belongs to $\mathcal{E}_{\nu,\nu+2} \setminus \mathcal{E}_{\nu-1,\nu+2}$.

We require two lemmas.

Lemma 12.2. *Let h be analytic on Δ and let $h(\mathbb{T}) \subset b\Gamma$. Then $h(\Delta) \subset \Gamma$ and so h is Γ -inner.*

Proof. Let $h = (s, p)$. Observe that $|s| \leq 2$ on Δ , by the Maximum Principle, and p is inner. We can suppose that $s^2 - 4p$ is not identically 0.

Let the zeros of $s^2 - 4p$ on \mathbb{T} be $\lambda_1, \dots, \lambda_N$ and let $h(\lambda_j) = (2\bar{\omega}_j, \bar{\omega}_j^2)$, $\omega_j \in \mathbb{T}$. Consider any $\omega \in \mathbb{T} \setminus \{\omega_1, \dots, \omega_N\}$. Note that Φ_ω is analytic on a neighbourhood of $\Gamma \setminus \{2\bar{\omega}, \bar{\omega}^2\}$.

For every $\lambda \in \mathbb{T} \setminus \{\lambda_1, \dots, \lambda_N\}$, we have $|s(\lambda)| < 2$. If $|s(\lambda)| = 2$ for $\lambda \in \mathbb{T}$ then $h(\lambda) \in \{(2\bar{\omega}_j, \bar{\omega}_j^2) : 1 \leq j \leq N\}$. Hence $2 - \omega s(\lambda) \neq 0$ for all $\lambda \in \mathbb{T}$. Therefore

$$\Phi_\omega \circ h = \frac{2\omega p - s}{2 - \omega s}$$

is analytic on Δ .

For any $\lambda \in \mathbb{T}$, $h(\lambda) \in b\Gamma$ and hence $|\Phi_\omega \circ h(\lambda)| = 1$. By the Maximum Principle, $|\Phi_\omega \circ h(\lambda)| \leq 1$ on Δ . Since this is true for all but finitely many $\omega \in \mathbb{T}$, by Proposition 3.2(2), $h(\lambda) \in \Gamma$.

Lemma 12.3. *Let $h = (s, p)$ be a rational Γ -inner function. Suppose that*

- (i) *h has N distinct royal nodes ω_j , $1 \leq j \leq N$, on \mathbb{T} , and*
- (ii) *there is a finite Blaschke product m of degree at most $\frac{1}{2}N$ such that $\Phi \circ (m, h)$ has cancellations at ω_j , $1 \leq j \leq N$.*

Then, for any $v \in \mathcal{B}l_{d(m)-1}$, the function $\Phi \circ (v, h)$ has no more than $d(m) + d(v)$ cancellations and the degree of $\Phi \circ (v, h)$ is at least $d(p) - d(m)$.

Proof. By Proposition 7.11, since there are cancellations in $\Phi \circ (m, h)$ at ω_j , $1 \leq j \leq N$, we have $m(\omega_j) = \frac{1}{2}\overline{s(\omega_j)}$, $1 \leq j \leq N$.

Suppose that, for some $v \in \mathcal{B}l_{d(m)-1}$, the function $\Phi \circ (v, h)$ has $d(m) + d(v) + 1$ cancellations. Then $v(\omega_j) = \frac{1}{2}\overline{s(\omega_j)}$ at $d(m) + d(v) + 1$ distinct points ω_j on \mathbb{T} . Hence the rational function $m - v$ vanishes at $d(m) + d(v) + 1$ distinct points ω_j on \mathbb{T} . The degree of $m - v$ is at most $d(m) + d(v)$ and so $m = v$. This is a contradiction to the assumption that $v \in \mathcal{B}l_{d(m)-1}$.

By Theorem 7.12, $\Phi \circ (v, h)$ has no double cancellations at ω_j . Thus $\Phi \circ (v, h)$ has degree at least $d(vp) - (d(m) + d(v)) = d(p) - d(m)$.

We can now prove Proposition 12.1.

Proof. It is clear that h_ν is analytic on Δ . Let $h_\nu = (s, p)$. It is simple to check that $s = \bar{s}p$ on \mathbb{T} , that $|s| \leq 2$ on \mathbb{T} and that $|s(\lambda)| = 2$ if and only if $\lambda^{2\nu+1} = -1$. For all $\lambda \in \mathbb{T}$,

$$|p(\lambda)| = \left| \frac{\lambda(\lambda^{2\nu+1} + r)}{1 + r\lambda^{2\nu+1}} \cdot \frac{1}{\bar{\lambda}^{2\nu+1}} \right| = \left| \frac{\lambda^{2\nu+1} + r}{\bar{\lambda}^{2\nu+1} + r} \right| = 1.$$

By Proposition 3.2, we have $h_\nu(\mathbb{T}) \subset b\Gamma$. Thus, by Lemma 12.2, h_ν is Γ -inner.

Let $m(\lambda) = -\lambda^\nu$, so that $m \in \mathcal{B}l_\nu$. It is simple to verify that

$$\Phi \circ (m, h_\nu) = \frac{2mp - s}{2 - ms}(\lambda) = -\lambda^{\nu+1} \in \mathcal{B}l_{\nu+1},$$

and so $h_\nu \in \mathcal{E}_{\nu, \nu+2}$.

To prove that h_ν is not in $\mathcal{E}_{\nu-1, \nu+2}$ we must show that, for all $v \in \mathcal{B}l_{\nu-1}$, the Blaschke product $\Phi \circ (v, h_\nu)$ has degree at least $\nu + 2$. By Proposition 7.11, for $v \in \mathcal{B}l_{\nu-1}$, if the function

$$\Phi \circ (v, h_\nu) = \frac{2vp - s}{2 - vs}$$

has a cancellation at ζ , then $\zeta \in \mathbb{T}$, ζ is a royal node for h_ν and $|s(\zeta)| = 2$. The royal nodes for h_ν , being the points at which $|s| = 2$, are the $(2\nu + 1)$ th roots of -1 , that is,

$$\omega_j = e^{i\pi(2j+1)/(2\nu+1)}, \quad j = 0, \dots, 2\nu.$$

Note that $s(\omega_j) = 2\omega_j^{\nu+1}$, $j = 0, \dots, 2\nu$. For the finite Blaschke product $m(\lambda) = -\lambda^\nu$, we have $m(\omega_j) = -\omega_j^\nu = \overline{\omega_j}^{\nu+1}$ since $\omega_j^{2\nu+1} = -1$. Hence $m(\omega_j) = \frac{1}{2}\overline{s(\omega_j)}$, $j = 0, \dots, 2\nu$. Thus $\Phi \circ (m, h)$ has cancellations at ω_j , $0 \leq j \leq 2\nu$. By Lemma 12.3, for every $v \in \mathcal{B}l_{\nu-1}$, the function $\Phi \circ (v, h_\nu)$ has no more than $d(v) + \nu$ different points of cancellation and the function $\Phi \circ (v, h_\nu)$ has degree at least $d(p) - d(m) = 2\nu + 2 - \nu = \nu + 2$. Therefore h_ν is not in $\mathcal{E}_{\nu-1, \nu+2}$.

Our main theorem follows easily.

Theorem 12.4. *For all $\nu \geq 1$, the condition \mathcal{C}_ν is strictly stronger than $\mathcal{C}_{\nu-1}$. In fact there is a set of Γ -interpolation data $\lambda_j \mapsto z_j$ with $\nu + 2$ interpolation points which satisfies $\mathcal{C}_{\nu-1}$ but not \mathcal{C}_ν .*

Proof. By Proposition 12.1, there exists a nonconstant function $h \in \mathcal{E}_{\nu, \nu+2} \setminus \mathcal{E}_{\nu-1, \nu+2}$. By Proposition 11.2, the condition \mathcal{C}_ν is strictly stronger than $\mathcal{C}_{\nu-1}$, and furthermore, there is a set of Γ -interpolation data $\lambda_j \mapsto z_j$ with $\nu + 2$ interpolation points which satisfies $\mathcal{C}_{\nu-1}$ but not \mathcal{C}_ν .

As we observed above, \mathcal{C}_0 is necessary and sufficient for solvability of a Γ -interpolation problem when $n = 2$, but a consequence of Theorem 12.4 is:

Corollary 12.5. *For all $n \geq 3$, Condition \mathcal{C}_{n-3} does not suffice for the solvability of an n -point Γ -interpolation problem.*

13. TABLE OF RELATIONS BETWEEN THE CLASSES $\mathcal{E}_{\nu k}$

The following table summarises the relations between \mathcal{E} -classes established above.

$$\begin{array}{cccccccccccc}
\mathcal{E}_{01} & \overset{(4,5)}{\subsetneq} & \mathcal{E}_{02} & \overset{(1)}{\subsetneq} & \mathcal{E}_{03} & \overset{(1)}{\subsetneq} & \mathcal{E}_{04} & \overset{(1)}{\subsetneq} & \mathcal{E}_{05} & \overset{(1)}{\subsetneq} & \mathcal{E}_{06} & \overset{(1)}{\subsetneq} & \mathcal{E}_{07} & \subsetneq \dots \\
\overset{(4)}{\parallel} & & \overset{(5)}{\parallel} & & \overset{(2)}{\not\cap} & & \cap & & \cap & & \overset{(3)}{\not\cap} & & \overset{(3)}{\not\cap} & \\
\mathcal{E}_{11} & \overset{(4,5)}{\subsetneq} & \mathcal{E}_{12} & \overset{(1)}{\subsetneq} & \mathcal{E}_{13} & \overset{(1)}{\subsetneq} & \mathcal{E}_{14} & \overset{(1)}{\subsetneq} & \mathcal{E}_{15} & \overset{(1)}{\subsetneq} & \mathcal{E}_{16} & \overset{(1)}{\subsetneq} & \mathcal{E}_{17} & \subsetneq \dots \\
\overset{(4)}{\parallel} & & \overset{(5)}{\parallel} & & \cap & & \overset{(2)}{\not\cap} & & \cap & & \cap & & \cap & \\
\mathcal{E}_{21} & \overset{(4,5)}{\subsetneq} & \mathcal{E}_{22} & \overset{(1)}{\subsetneq} & \mathcal{E}_{23} & \overset{(1)}{\subsetneq} & \mathcal{E}_{24} & \overset{(1)}{\subsetneq} & \mathcal{E}_{25} & \overset{(1)}{\subsetneq} & \mathcal{E}_{26} & \overset{(1)}{\subsetneq} & \mathcal{E}_{27} & \subsetneq \dots \\
\overset{(4)}{\parallel} & & \overset{(5)}{\parallel} & & \cap & & \cap & & \overset{(2)}{\not\cap} & & \cap & & \cap & \\
\mathcal{E}_{31} & \overset{(4,5)}{\subsetneq} & \mathcal{E}_{32} & \subset & \mathcal{E}_{33} & \overset{(1)}{\subsetneq} & \mathcal{E}_{34} & \overset{(1)}{\subsetneq} & \mathcal{E}_{35} & \overset{(1)}{\subsetneq} & \mathcal{E}_{36} & \overset{(1)}{\subsetneq} & \mathcal{E}_{37} & \subsetneq \dots \\
\overset{(4)}{\parallel} & & \overset{(5)}{\parallel} & & \cap & & \cap & & \cap & & \overset{(2)}{\not\cap} & & \cap & \\
\mathcal{E}_{41} & \overset{(4,5)}{\subsetneq} & \mathcal{E}_{42} & \subset & \mathcal{E}_{43} & \subset & \mathcal{E}_{44} & \overset{(1)}{\subsetneq} & \mathcal{E}_{45} & \overset{(1)}{\subsetneq} & \mathcal{E}_{46} & \overset{(1)}{\subsetneq} & \mathcal{E}_{47} & \subsetneq \dots \\
\overset{(4)}{\parallel} & & \overset{(5)}{\parallel} & & \cap & & \cap & & \cap & & \cap & & \overset{(2)}{\not\cap} & \\
\dots & & \dots & & \dots & & \dots & & \dots & & \dots & & \dots & \\
\mathcal{E}_{\nu 1} & \overset{(4,5)}{\subsetneq} & \mathcal{E}_{\nu 2} & \subset & \mathcal{E}_{\nu 3} & \subset & \mathcal{E}_{\nu 4} & \subset & \mathcal{E}_{\nu 5} & \subset & \mathcal{E}_{\nu 6} & \subset & \mathcal{E}_{\nu 7} & \subset \dots \\
\overset{(4)}{\parallel} & & \overset{(5)}{\parallel} & & \cap & & \cap & & \cap & & \cap & & \cap & \\
\dots & & \dots & & \dots & & \dots & & \dots & & \dots & & \dots &
\end{array}$$

Remark 13.1. (1) In Corollary 7.14 we proved that, for all $\nu \geq 0$, every rational Γ -inner function h of the form $(0, p)$ is in $\mathcal{E}_{\nu, \nu+d(p)+1} \setminus \mathcal{E}_{\nu, \nu+d(p)}$. In Example 7.16, for a rational inner function ψ on \mathbb{D} , we considered the rational Γ -inner function

$$h_\psi(\lambda) = (\lambda + \lambda\psi(\lambda), \lambda^2\psi(\lambda)), \quad \lambda \in \mathbb{D},$$

and proved that $h_\psi \in \mathcal{E}_{1, d(\psi)+2} \setminus \mathcal{E}_{1, d(\psi)+1}$.

(2) In Proposition 12.1, for all $\nu \geq 1$, we presented a function $h_\nu \in \mathcal{E}_{\nu, \nu+2} \setminus \mathcal{E}_{\nu-1, \nu+2}$.

(3) In Example 7.17, for $j = 1, 2, \dots$, we constructed a rational function $h_j \in \mathcal{E}_{1, 2j+4} \setminus \mathcal{E}_{0, 2j+4}$.

(4) In Section 8 we showed that, for every $\nu \geq 0$, the class $\mathcal{E}_{\nu 1}$ consists of the superficial rational Γ -inner functions.

(5) In Section 10 we proved that, for every $\nu \geq 0$, the class $\mathcal{E}_{\nu 2}$ comprises precisely the complex geodesics of \mathbb{G} and the superficial rational Γ -inner functions.

(6) In Theorem 9.1 we showed that, for $\nu \geq 0$ and $k \geq 2$, functions in $\mathcal{E}_{\nu k}$ are either superficial or k -extremal.

Remark 13.2. If the Γ -interpolation Conjecture is true then, for $k \geq 3$, the columns are all ultimately constant:

$$\mathcal{E}_{k-2, k} = \mathcal{E}_{k-1, k} = \mathcal{E}_{kk} = \mathcal{E}_{k+1, k} = \dots$$

By Theorem 9.1, if $h \in \mathcal{E}_{\nu, k}$ then either h is superficial, in which case h belongs to all $\mathcal{E}_{\nu n}$, or h is k -extremal, and then, by Observation 9.2, $h \in \mathcal{E}_{k-2, k}$, providing that Conjecture 4.1 holds.

14. CONCLUDING REFLECTIONS

Study of the interpolation problem for $\text{Hol}(\mathbb{D}, \Gamma)$ was originally motivated by a wish to solve the “ μ -synthesis problem”, which arises in control engineering [16, 17]. This is a hard problem of a function-theoretic nature, and its solution would have considerable significance for engineers. Unfortunately, at present it can be analysed in only a few very special cases [33]; in this paper we throw some light on a further case – the *spectral Nevanlinna-Pick problem for 2×2 matrix functions* with $n > 2$ interpolation points. Given points $\lambda_1, \dots, \lambda_n \in \mathbb{D}$ and target matrices $W_1, \dots, W_n \in \mathbb{C}^{2 \times 2}$ one seeks an analytic 2×2 -matrix-valued function F such that

$$F(\lambda_j) = W_j \quad \text{for } j = 1, \dots, n, \text{ and}$$

$$r(F(\lambda)) \leq 1 \quad \text{for all } \lambda \in \mathbb{D}$$

where r denotes the spectral radius. This problem is essentially equivalent to the interpolation problem for $\text{Hol}(\mathbb{D}, \Gamma)$ studied here; see [5, Theorem 1.1].

If Conjecture 4.1 is true then one can check¹ whether a given spectral Nevanlinna-Pick problem $\lambda_j \mapsto W_j$, $1 \leq j \leq n$, has a solution by determining whether the Γ -interpolation data $\lambda_j \mapsto (\text{tr } W_j, \det W_j)$, $1 \leq j \leq n$, satisfy condition \mathcal{C}_{n-2} (where $n \geq 2$). To verify condition \mathcal{C}_{n-2} one must check for positivity a pencil of $n \times n$ matrices indexed by \mathcal{Bl}_{n-2} , the set of Blaschke products of degree at most $n-2$. Now \mathcal{Bl}_{n-2} is a compact set of real dimension $2n-3$ in the topology of locally uniform convergence. In cases of engineering interest n is likely to be small, and so there is a fair prospect that condition \mathcal{C}_{n-2} can be checked efficiently. We have not attempted any numerical studies. Engineers currently use a heuristic algorithm called “D-K iteration” [17, Section 9.3.3], based on results of Bercovici, Foias and Tannenbaum [9, 10], to attempt to solve μ -synthesis problems, but this algorithm is slow and unreliable. For the n -point Γ -interpolation problem it requires a search over an unbounded, nonconvex set of $6n$ real dimensions. At least for this very special case of μ -synthesis, if the Γ -interpolation Conjecture is true then one should be able to improve substantially on current methods.

Finding good algorithms is one goal of our research, but equally important is to develop a satisfactory analytic theory of μ -synthesis problems. For example, in proving the Γ -interpolation Conjecture one might be able to show that a solvable n -point Γ -interpolation problem has a solution that is Γ -inner of degree at most $2n-2$. Aside from its theoretical interest, such a result could have practical applications. A good analytic theory would explain the phenomenon of ill-conditioning which engineers have encountered, and would enable numerical analysts to test their algorithms against a range of examples that are exactly solvable. One could also hope to derive parametrizations of solution sets of a range of μ -synthesis problems, like those that exist for classical Nevanlinna-Pick problems.

We finish with some questions whose answers would be significant for the understanding of μ -synthesis. The main question we leave open is of course whether the Γ -interpolation Conjecture is true, but here are two more.

Question 14.1. *Is there a bang-bang theorem for n -extremals?*

¹subject to a minor complication in the case that some W_j is a scalar matrix.

That is, if h is n -extremal for $\text{Hol}(\Omega_1, \Omega_2)$, then does h necessarily map the topological or distinguished boundary of Ω_1 into the corresponding boundary of Ω_2 ? We are particularly interested in the question in the case of $\text{Hol}(\mathbb{D}, \mathbb{G})$. There is a general bang-bang theorem due to J. W. Helton and R. Howe which can be applied to $\text{Hol}(\mathbb{D}, \Omega)$ [19], but it assumes that Ω has a smooth boundary, and so does not apply to \mathbb{G} .

Question 14.2. *Is every n -extremal in $\text{Hol}(\mathbb{D}, \Gamma)$ rational?*

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